EX: The gain of an amplifier circuit (the ratio of the output voltage amplitude to the input voltage amplitude) typically has the following behavior:

1. The gain starts at zero for DC (or frequency = 0Hz) in order to eliminate any constant offset that might cause undesirable effects (like making speakers pop). This behavior is accomplished by putting a capacitor in series with the input signal.

2. The gain rises as frequency increases until leveling off at some constant value that we call the midband gain.

3. As the frequency continues to rise, the gain starts to fall off again at some higher frequency owing to small capacitances in the circuit starting to short out.

One way to find the frequency response or transfer function, $H(j\omega)$, of a circuit is by transforming to the frequency domain, using phasors and impedances written in terms of $j\omega$, and calculating the magnitude and phase of phasor V_{out} divided by phasor V_{in}.

$$H(j\omega) = \frac{V_{\text{out}}}{V_{\text{in}}}$$

Such a calculation, however, is tedious, is hard to verify, and is opaque in terms of insight into how component values affect the shape of the magnitude-gain curve.

To address the shortcomings of analysis using $j\omega$, we employ the concept of "poles" and "zeros". The theory behind this is the Laplace transform, which is covered in other courses. Here, we just use a key result of the theory, that makes sense on its own, although it is difficult to imagine how it would be developed by itself. That key result is that we use $s = j\omega$ to express the transfer function of a circuit. That is, we use a transfer function in terms of s.

$$H(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)}$$

Next, we do some algebra to express $H(s)$ as a polynomial in s over another polynomial in s. Then we factor the polynomials in terms of s to express them as products of root terms in s.

Here, we consider the following example of an $H(s)$ that has been derived for an amplifier circuit using impedances, $Z_C = \dfrac{1}{sC}$ and $Z_L = sL$:

$$H(j\omega) = H(s) = \frac{s^2 + 110s + 1000}{s^3 + 1050s^2 + 50000s}$$

We now factor the polynomials into root terms:

$$H(s) = \frac{(s+10)(s+100)}{s(s+50)(s+1000)}.$$

The roots of these polynomial terms will be negative numbers (or complex numbers with negative real parts) for our linear circuits. We refer to the **roots of the numerator** as **zeros,** $z_1 = -10$, $z_2 = -100$ and the **roots of the denominator** as **poles**, $p_1 = 0$, $p_2 = -50$, $p_3 = -1000$. Note that the roots may be real numbers for s, which means the roots would correspond to *imaginary* frequencies, ω. This is a strange idea! Nonetheless, using s makes analysis much easier, as we now employ a remarkable sequence of ideas that mean the hard work is done once we have determined the poles and zeros.

TOOL: The Bode plot introduces approximations and scaling that allow us to make quick sketches of $|H(s)|$ and $\angle H(s)$.

In summary, the Bode magnitude plot rules for a real zero or pole at ω_0 involve approximating the zero or pole term at low frequency as $\left|\dfrac{s}{\omega_0} + 1\right| = \left|\dfrac{j\omega}{\omega_0} + 1\right| \approx 1$ for $\omega << \omega_0$ and at high frequency as $\left|\dfrac{s}{\omega_0} + 1\right| = \left|\dfrac{j\omega}{\omega_0} + 1\right| \approx \left|\dfrac{\omega}{\omega_0}\right| = \dfrac{\omega}{\omega_0}$ for $\omega >> \omega_0$. We then convert the vertical axis to dB ($20\log_{10}$) and the horizontal axis to a log scale.

DERIV: In reality, our $H(s)$ is complex, given that $s = j\omega$, but we want to plot this complex $H(s)$ versus ω. The solution is to make two plots, one for the magnitude, $|H(s)|$, and one for the phase angle, $\angle H(s)$. In other words, we are plotting the polar form of $H(s)$. The polar form of $H(s)$ corresponds to what is displayed on an oscilloscope and is vastly preferable to an uninformative plot of the rectangular form of $H(s)$.

For the magnitude of $H(s)$. we exploit the property that the magnitude of a product is the product of magnitudes. Thus, the magnitude or gain of our example transfer function may be written as a product of magnitudes:

$$|H(j\omega)| = |H(s)| = \frac{|s+10| \cdot |s+100|}{|s| \cdot |s+50| \cdot |s+1000|}.$$

For the magnitude Bode plot of $H(s)$, we carry out the following steps:

- We translate the root terms into unitless terms by factoring out poles and zeros:

$$|H(s)| = \frac{10\cdot 100\cdot\left|\frac{s}{10}+1\right|\left|\frac{s}{100}+1\right|}{1\cdot 50\cdot 1000\cdot\left|\frac{s}{1}\right|\left|\frac{s}{50}+1\right|\left|\frac{s}{1000}+1\right|} = \frac{1}{50}\frac{\left|\frac{s}{10}+1\right|\left|\frac{s}{100}+1\right|}{\left|\frac{s}{1}+1\right|\left|\frac{s}{50}+1\right|\left|\frac{s}{1000}+1\right|}$$

- We take transform each term into its decibel or dB value. This means we are taking 20 times the log (base 10) of each quantity, which turns the product into a sum.

$$H(s) = 20\log_{10}\frac{1}{50} + 20\log_{10}\left|\frac{s}{10}+1\right| + 20\log_{10}\left|\frac{s}{100}+1\right|$$
$$-20\log_{10}|s| - 20\log_{10}\left|\frac{s}{50}+1\right| - 20\log_{10}\left|\frac{s}{1000}+1\right|$$

- We plot each term as straight lines on a plot of dB vs log ω. The exact curve is well-approximated by the straight lines that meet at $\omega = \omega_0$:
  $20\log_{10} 1 = 0$ for $\omega < \omega_0$, a horizontal line at height 0 dB
  $20\cdot(\log_{10}\omega - \log_{10}\omega_0)$ for $\omega > \omega_0$, a line sloping up at 20 dB/decade

- We sum the plots of the individual terms to obtain the overall Bode magnitude plot.

The result for our example is shown in Fig 1. Note that pole and zero frequencies are shifted down by a factor of 2π when plotting frequency vs Hz rather than rad/s.

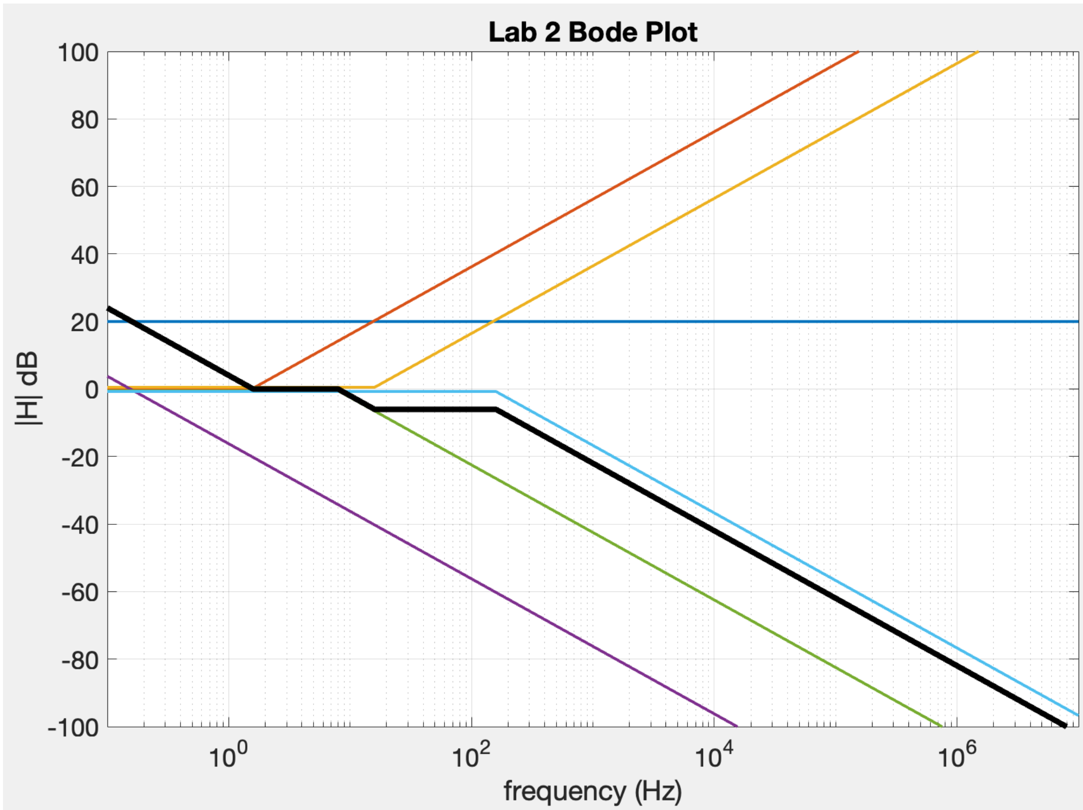

Fig. 1. Bode magnitude plot for example transfer function.