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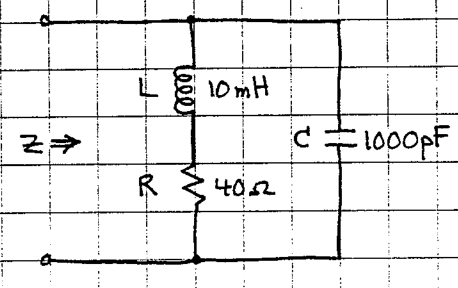

Plot $|z|$ and determine how $|z|$ depends (qualitatively) on R, L, and C.

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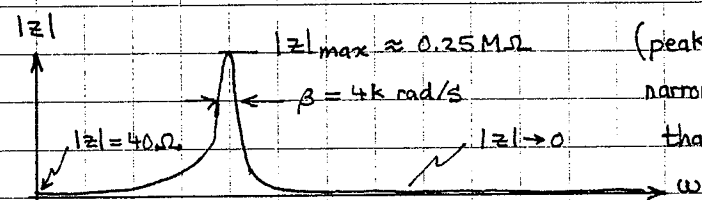
ans: (plot)



(peak is actually much narrower and sharper than shown here)

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soln: $z = (z_L + z_R) \parallel z_C = (j\omega L + R) \parallel \dfrac{1}{j\omega C}$

$$" = \frac{\dfrac{j\omega L + R}{j\omega C}}{j\omega L + R + \dfrac{1}{j\omega C}} = \frac{\dfrac{L}{C} + \dfrac{R}{j\omega C}}{R + j\left(\omega L - \dfrac{1}{\omega C}\right)}$$

$$" = \frac{\dfrac{L}{C}\left(1 + \dfrac{R}{j\omega L}\right)}{R + j\left(\omega L - \dfrac{1}{\omega C}\right)} = \frac{L}{RC}\,\frac{\left(1 - j\dfrac{R}{L}\dfrac{1}{\omega}\right)}{1 + j\left(\omega \dfrac{L}{R} - \dfrac{1}{\omega RC}\right)}$$

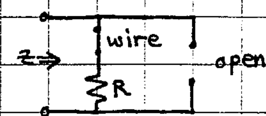
The last expression for z is convenient for several reasons:

1) The numerator and denominator are both of form $1 \pm jW$ where W is a real quantity. In other words, the real part is one, and we can compare the magnitude of the real and imaginary parts by comparing 1 and W.
2) ω only appears in the imaginary parts of the numerator and denominator.
3) All the coefficients multiplying ω are time constants.
4) The scaling factor of $\dfrac{L}{RC}$ out front gives us some information about the dependence of z on each component value.

Before proceeding further, we observe the following:

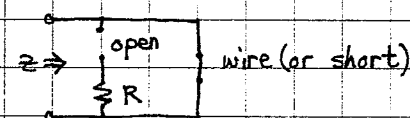
1) For $\omega = 0$ $\quad j\omega L = 0$ and $1/j\omega C = \infty$

Thus, we have
$z = R = 40\,\Omega$

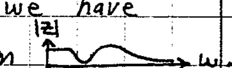


2) For $\omega \to \infty$ $\quad j\omega L = \infty$ and $1/j\omega C = 0$

Thus, we have
$z = 0\,\Omega$



3) Since we have an RLC circuit, we might get a resonant peak or dip at the resonant frequency
$\omega_0 = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{10\text{m}\ 1000\text{p}}} = \dfrac{1}{\sqrt{10}}$ M rad/s.

We expect a peak (rather than a dip) because we already know that $z \to 0$ as $\omega \to \infty$, and we have never seen an RLC response vs ω of form .
In other words, $|z|$ would have too many wiggles.

At $\omega_0 = \dfrac{1}{\sqrt{LC}}$, we have $\text{Im}[\text{Denominator of } z] = 0$.

This condition helps to make z larger. <u>But</u> we also have an ω dependence in the numerator that may shift the frequency where the max of $|z|$ occurs.

To find ω_p where $\max\limits_{\omega} |z|$ occurs, we could use $\dfrac{d}{d\omega}|z| = 0$ at ω_p. Then we could solve for ω_p.

There is a simpler approach, however. We may be able to argue that $\omega_p \approx \omega_0$. If so, we save a lot of work!

We will need to know the following values in our upcoming calculations and approximations:

$$\frac{L}{R} = \frac{10m}{40} = \frac{1}{4}\,ms \quad \& \quad \frac{R}{L} = 4k/s$$

$$RC = 40 \cdot 1kp = 40\,ns \quad \& \quad \frac{1}{RC} = \frac{1}{40}\,G/s = 25M/s$$

$$\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10m \cdot 1kp}} = \frac{1M}{\sqrt{10}}\,rad/s \quad \& \quad \sqrt{LC} = \sqrt{10}\,\mu s = \frac{1}{\omega_o}$$

Now consider $\omega = \omega_o$. We have:

$$z\Big|_{\omega_o} = \frac{\frac{L}{RC}\left(1 - j\frac{R}{L}\frac{1}{\omega_o}\right)}{1 + j \cdot 0} = \frac{L}{RC}\left(1 - j\frac{R}{L}\frac{1}{\omega_o}\right)$$

$$= \frac{L}{RC} - j\frac{1}{C}\frac{1}{\omega_o} = \frac{1}{C}\left(\frac{L}{R} - \frac{j}{\omega_o}\right)$$

$$= \frac{1}{C}\left(\frac{1}{4}ms - j\underbrace{\sqrt{10}\,\mu s}_{\text{much smaller than real part}}\right)$$

$$\approx \frac{1}{C}\,\frac{1}{4}ms = \frac{1}{1kp}\,\frac{1}{4}ms = \frac{1}{4}\,M\Omega$$

Note how the imaginary part was small and could be ignored. This suggests that, when we look for the ω where $\max_{\omega}|z|$ occurs, the imaginary part of the numerator probably doesn't move the ω_p for max $|z|$ very far from $\omega_o = 1/\sqrt{LC}$.

To prove $\omega_p \approx \omega_o$, we show that a small change in ω, when $\omega \approx \omega_o$, will cause a large increase in |denominator of z| and only a small increase in |numerator of z|. Thus, $|z|$ would go down for $\omega \neq \omega_o$.

At ω_0, $\dfrac{R}{L}\dfrac{1}{\omega} = \dfrac{4\text{ k/s}}{\frac{1}{\sqrt{10}}\text{ M/s}} = 4\sqrt{10}\text{ m} \ll 1,$

and $\dfrac{L}{R}\omega = \dfrac{1\text{k}}{4\sqrt{10}} \gg 1,$

and $\dfrac{1}{\omega RC} = \dfrac{1}{\frac{1}{\sqrt{10}}\text{ M/s } 40 \text{ } 1\text{ kp}} = \dfrac{\sqrt{10}}{40}\text{ 1k} = \dfrac{1\text{k}}{4\sqrt{10}} \gg 1.$

To show that the magnitude of the denominator of $|z|$ changes faster than the numerator of $|z|$, we use derivatives.

$$|z| = \frac{L}{RC}\frac{\left|1 - j\frac{R}{L}\frac{1}{\omega}\right|}{\left|1 + j\left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)\right|} = \frac{L}{RC}\frac{\sqrt{1^2 + \left(\frac{R}{L}\frac{1}{\omega}\right)^2}}{\sqrt{1^2 + \left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)^2}}$$

Numerator: $\dfrac{d}{d\omega}\dfrac{R}{L}\dfrac{1}{\omega}\Bigg|_{\omega_0} = \dfrac{R}{L}\left(\dfrac{-1}{\omega^2}\right)\Bigg|_{\omega_0} = \dfrac{40}{10\text{m}}\dfrac{-1}{\left(\frac{1\text{M}}{\sqrt{10}}\right)^2} = -40\text{ ns}$

Denominator: $\dfrac{d}{d\omega}\left(\omega\dfrac{L}{R} - \dfrac{1}{\omega RC}\right)\Bigg|_{\omega_0} = \dfrac{L}{R} + \dfrac{1}{RC\omega^2}\Bigg|_{\omega_0}$

$$= \frac{10\text{M}}{40} + \frac{1}{40\text{n}\left(\frac{1\text{M}}{\sqrt{10}}\right)^2} = \frac{1}{4}\text{m} + \frac{1}{4}\text{m s}$$

$$= \frac{1}{2}\text{ ms}$$

The magnitude of the denominator change for a small change in ω is clearly much larger than the magnitude of the numerator change. We conclude that moving away from ω_0 will lower $|z|$. $\therefore\ \omega_p \approx \omega_0$

Note: The minus sign in numerator derivative just means the numerator increases if ω decreases.

From the above discussion, we conclude that we may approximate z for $\omega \approx \omega_0$ as follows:

$$z\Big|_{\omega\approx\omega_0} \approx \frac{L/RC}{1+j\left(\omega\frac{L}{R}-\frac{1}{\omega RC}\right)}$$

Now we can find the "bandwidth" of the peak in $|z|$.

We solve $|z| = \frac{1}{\sqrt{2}}\cdot \underbrace{|z|\Big|_{\omega=\omega_0}}_{\text{i.e. } \max_\omega |z| \text{ (approximately)}} = \frac{1}{\sqrt{2}}\, L/RC$

$$|z| \approx \frac{L/RC}{\sqrt{1^2+\left(\omega\frac{L}{R}-\frac{1}{\omega RC}\right)^2}} = \frac{1}{\sqrt{2}}\frac{L}{RC}$$

or $\quad 1^2+\left(\omega\frac{L}{R}-\frac{1}{\omega RC}\right)^2 = 2$

or $\quad \left(\omega\frac{L}{R}-\frac{1}{\omega RC}\right)^2 = 1^2$

or $\quad \omega\frac{L}{R}-\frac{1}{\omega RC} = \pm 1$

or $\quad \omega^2\frac{L}{R}-\frac{1}{RC} = \pm\omega$

or $\quad \omega^2 \pm \frac{R}{L}\omega - \frac{1}{LC} = 0$

$$\omega_{c1,2} = \mp\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2+\frac{1}{LC}}$$

and $\quad \beta \doteq \omega_{c2}-\omega_{c1} = \frac{R}{L} = \frac{40}{10m} = 4k \text{ rad/s}$

Summary: $\quad \max_\omega |z| \approx \frac{L}{RC} \qquad \beta \approx \frac{R}{L}$

$$|z|\Big|_{\omega=0} = R \qquad |z|\Big|_{\omega\to\infty} = 0\,\Omega$$