Padé Approximations

Suppose we have an analytic function so that it may be written as

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

We approximate $f(x)$ by using a ratio of polynomials: (Note that $g_0 = 1$)

$$f(x) \approx \frac{p(x)}{g(x)} = \frac{p_0 + p_1 x + p_2 x^2 + \dots p_L x^L}{1 + g_1 x + g_2 x^2 + \dots + g_M x^M}$$

This is the Padé approximation $R_{L/M}$ or $[L/M]$ when the coeff's. of $p(x)$ and $g(x)$ are chosen to make $f(x) - \frac{p(x)}{g(x)} = O(x^{L+M+1})$ as $x \to 0$.

To find coeff's for $p(x)$ and $g(x)$, we solve the following equation by equating coeff's of terms of the same order:

$$g(x) f(x) = p(x)$$

We have the following product for $g(x) f(x)$:

$$g(x) f(x) = (1 + g_1 x + g_2 x^2 + \dots + g_M x^M)$$
$$(a_0 + a_1 x + a_2 x^2 + \dots + a_N x^N + \dots)$$
$$= a_0 + (a_0 g_1 + a_1) x + (a_0 g_2 + a_1 g_1 + a_2) x^2$$
$$+ (a_0 g_3 + a_1 g_2 + a_2 g_1 + a_3) x^3$$
$$+ \dots$$

Equating coeff's for terms of the same order, (i.e., same power of x), in $p(x)$ yields equations:

$$a_0 = p_0$$

$$a_0 g_1 + a_1 = p_1$$

$$a_0 g_2 + a_1 g_1 + a_2 = p_2$$

$$\vdots$$

$$a_0 g_L + a_1 g_{L-1} + \dots + a_L = p_L$$

At this point, we have exhausted the coeff's of $p(x)$ but may still have coeff's of $g(x)f(x)$. We use zeros for the coeffs of $p(x)$ from this point on:

$$a_0 g_{L+1} + a_1 g_L + \dots + a_{L+1} = 0$$

$$\vdots$$

$$a_0 g_M + a_1 g_{M-1} + \dots + a_M = 0$$

At this point, we run out of g_M's coeff's for $g(x)$ and have to use $a_{k>0}$ with g_M to get the coefficient of $x^{n>M}$ in $g(x)f(x)$:

$$a_1 g_M + a_2 g_{M-1} + \dots + a_{M+1} = 0$$

$$\vdots$$

$$a_L g_M + a_{L+1} g_{M-1} + \dots + a_{M+L} = 0$$

At this point, we have all the equations to be solved.

Writing the equations in matrix form, we have

$$\begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots \\ a_L & a_{L-1} & a_{L-2} & \cdots a_0 & 0\cdots \\ \hline a_{L+1} & a_L & a_{L-1} & \cdots a_0 & 0\cdots \\ \vdots \\ a_M & a_{M-1} & a_{M-2} & \cdots & a_0 \\ a_{M+1} & a_M & a_{M-1} & \cdots & a_1 \\ \vdots \\ a_{M+L} & a_{M+L-1} & a_{M+L-2} & \cdots & a_L \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ q_2 \\ \vdots \\ q_M \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_L \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

We can solve these equations, first for q's, then for p's. We set the q values by using the equations below the dashed line and moving the first term to the other side of the equation:

$$\underbrace{\begin{bmatrix} a_L & a_{L-1} & \cdots & a_0 & 0 \cdots \\ \vdots \\ a_{M-1} & a_{M-2} & \cdots & a_1 & a_0 \\ a_M & a_{M-1} & \cdots & a_2 & a_1 \\ \vdots \\ a_{M+L-1} & a_{M+L-2} & \cdots & a_{L+1} & a_L \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \end{bmatrix}}_{\cdot\ \vec{q}} = \underbrace{\begin{bmatrix} -a_{L+1} \\ \vdots \\ -a_M \\ -a_{M+1} \\ \vdots \\ -a_{M+L} \end{bmatrix}}_{=\ \vec{a}}$$

or

$$A \vec{q} = \vec{a}$$

We have the solution

$$\vec{q} = A^{-1} \vec{a}.$$

Once we have q values, we use the equations above the dashed lines to find p's.

ex: Find the $R_{2/2} \equiv [2,2]$ approximation for e^x, (denoted as $\exp_{2/2}(x)$).

We start with the power series (or Taylor series) expansion for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

So $a_0 = 1$, $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{6}$, $a_4 = \frac{1}{24}$, ...

Our equations are

$$\begin{bmatrix} a_0 & 0 & 0 \\ a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \\ \hdashline a_3 & a_2 & a_1 \\ a_4 & a_3 & a_2 \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ 0 \\ 0 \end{bmatrix}$$

For the equation below the dashed line, we have

$$\begin{bmatrix} a_2 & a_1 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} -a_3 \\ -a_4 \end{bmatrix}$$

Using values, we have

$$\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{6} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{24} \end{bmatrix}$$

$$A \qquad \vec{q} \qquad = \qquad \vec{a}$$

Using $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

we have $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \dfrac{1}{\frac{1}{4} - \frac{1}{6}} \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{24} \end{bmatrix}$

$$'' = 12 \begin{bmatrix} -\frac{1}{12} + \frac{1}{24} \\ \frac{1}{36} - \frac{1}{48} \end{bmatrix}$$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{12} \end{bmatrix}$$

Now we find the p's:

$$\begin{bmatrix} a_0 & 0 & 0 \\ a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{12} \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}$$

$p_0 = 1$

$p_1 = \frac{1}{2}$

$p_2 = \frac{1}{12}$

Thus, $\exp_{2/2}(x) = \dfrac{1 + \frac{1}{2}x + \frac{1}{12}x^2}{1 - \frac{1}{2}x + \frac{1}{12}x^2}$

Comparison of values:

x	e^x	$\exp_{2/2}$

x	e^s	exp 2/2	err	err %
0	1	1	0	0
0.1	1.10517092	1.1051709	-1.5359E-08	-1.3897E-06
0.2	1.22140276	1.22140221	-5.4414E-07	-4.455E-05
0.3	1.34985881	1.34985423	-4.5802E-06	-0.00033931
0.4	1.4918247	1.49180328	-2.1419E-05	-0.00143576
0.5	1.64872127	1.64864865	-7.2622E-05	-0.00440475
0.6	1.8221188	1.82191781	-0.00020099	-0.01103068
0.7	2.01375271	2.013269	-0.00048371	-0.02402026
0.8	2.22554093	2.2244898	-0.00105113	-0.04723043
0.9	2.45960311	2.45748988	-0.00211323	-0.08591763
1	2.71828183	2.71428571	-0.00399611	-0.14700882
1.1	3.00416602	2.99697428	-0.00719174	-0.23939231
1.2	3.32011692	3.30769231	-0.01242462	-0.37422221
1.3	3.66929667	3.64855688	-0.02073979	-0.56522526
1.4	4.05519997	4.02158273	-0.03361723	-0.82899076
1.5	4.48168907	4.42857143	-0.05311764	-1.18521479
1.6	4.95303242	4.87096774	-0.08206468	-1.65685736
1.7	5.47394739	5.34968017	-0.12426722	-2.27015739
1.8	6.04964746	5.86486486	-0.1847826	-3.05443583
1.9	6.68589444	6.41567696	-0.27021748	-4.04160558
2	7.3890561	7	-0.3890561	-5.26530173