Approximation Theory —

23 May 1990 Stone - Weierstrass Theorem examples — Fourier series

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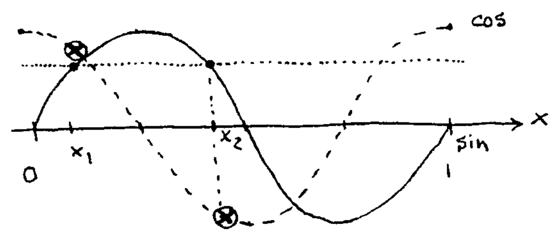
ex: Fourier series satisfy S-W thm:

$$\mathcal{F} = \{ f(x) = \sum_{n=0}^{\infty} a_n \sin(2\pi n x) + b_n \cos(2\pi n x) \qquad a_n, b_n \in \mathbb{R} ;\ x \in [0,1] \}$$

I) Identity $\quad f(x) = 1 \cos(2\pi 0 x) = 1 \in \mathcal{F}$ ✓

II) Separability $\quad$ Consider $\sin(2\pi x)$ and $\cos(2\pi x)$



We observe that when $x_1 \neq x_2$ then if $\cos(2\pi x_1) = \cos(2\pi x_2)$ (i.e. the intersection of dotted line and cos) we see that $\sin(2\pi x_1) \neq \sin(2\pi x_2)$ (i.e. ⊗ points).

The only exception to this rule is $x_1 = 0$, $x_2 = 1$. Indeed, 0 and 1 are <u>not</u> separable. Thus, we have a problem. We can fix this problem by working an a slightly smaller interval, $[0, 1-\epsilon]$ ϵ small. Since we can get an interval arbitrarily close to $[0,1]$ we can also use the entire interval $[0,1]$ with the understanding that we can only approximate funcs, $g(x)$, such that $g(0) = g(1)$. We cannot approximate a func having different values on the end points. Nevertheless, we fail to approximate such funcs <u>only</u> at the endpoints.

III) Closure

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(2\pi n x) + b_n \cos(2\pi n x) \qquad g(x) = \sum_{m=0}^{\infty} c_m \sin(2\pi m x) + d_m \cos(\ldots$$

$$a f(x) + b g(x) = \sum_{n=0}^{\infty} (a_n + c_n) \sin(2\pi n x) + (b_n + d_n) \cos(2\pi n x) \in \mathcal{F} \ ✓$$

$$f(x) g(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n c_m \sin(2\pi n x) \sin(2\pi m x) + \text{etc.}$$

Can expand $\Sigma\Sigma$ as single Σ, sin·sin as sum of sin, cos ✓