Approximation Theory —

21 May 1990 Stone - Weierstrass Theorem ~ Examples — Polynomials

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ex: Polynomials in x satisfy the Stone-Weierstrass thm:

$$\mathcal{F} = \{ f(x) : f(x) = \sum_{n=0}^{\infty} a_n x^n , \quad a_n \in \mathbb{R} \} \qquad \mathbb{R} = \text{Real \#'s}$$

I) Identity function $f(x) = 1 \cdot x^0 + 0 \cdot x^1 + 0 \cdot x^2 + \ldots = 1 \in \mathcal{F}$ ✓

II) Separability $f(x) = 0 \cdot x^0 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \ldots = x \in \mathcal{F}$

$$f(x_1) \neq f(x_2) \quad \text{when} \quad x_1 \neq x_2$$ ✓

III) Algebraic Closure $f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad g(x) = \sum_{m=0}^{\infty} b_m x^m$

$$af(x) + bg(x) = \sum_{n=0}^{\infty} (aa_n + bb_n)\, x^n \in \mathcal{F}$$ ✓

$$\begin{aligned} f(x)\,g(x) &= \left(\sum_{n=0}^{\infty} a_n x^n \right)\left(\sum_{m=0}^{\infty} b_m x^m \right) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m x^{n+m} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n+m=k} a_n b_m \right) x^k \\ &= \sum_{k=0}^{\infty} c_k x^k \in \mathcal{F} \quad \text{where } c_k = \sum_{n+m=k} a_n b_m \end{aligned}$$

Many approximation methods are based on polynomials. Familar examples are Legendre, Chebyshev, and Bernoulli polynomials. One might say polynomials are the prototypical example of the Stone-Weierstrass theorem since the theorem says, in essence, that all continuous functions are polynomials.