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$\mathcal{G}$ is a basis for $\mathcal{F}$ $\equiv$ 1) $\mathcal{G}$ is a set of elements $\{g_n\}$ from $\mathcal{F}$

2) $\mathcal{F}$ is a linear (or equivalently, a vector) space

and for each $f \in \mathcal{F}$ there is a unique set of coefficients $\{a_n\} \in$ field $\mathcal{A}$ such that we can express f as

$$f = \sum_{n=1}^{\infty} a_n g_n .$$

Note: series coeff's come from field $\mathcal{A}$, whereas $\mathcal{F}$ is a linear space or some possibly different space $\mathcal{B}$.

In other words, every function (or element) in $\mathcal{F}$ can be written as a weighted sum of basis functions.

Technically, we replace the above equality with

$$\lim_{n \to \infty} \| f - \sum_{i=1}^{n} a_i g_i \| .$$

This means we only have to consider the set of all finite series, with the closure of this set being $\mathcal{F}$.

ex: $\mathcal{G} = \{[1,0], [0,1]\}$ is a basis for $\mathcal{F} = 2 \times 1$ complex vectors for the field of complex numbers $\mathcal{A} = \mathbb{C}$.

pf: We have to show that we can write any vector $[c_1, c_2]$ with complex ($\in \mathbb{C}$) entries in the form $a_1 \cdot [1,0] + a_2 \cdot [0,1]$ with $a_1, a_2 \in \mathbb{C}$. But this is simple; we just use $a_1 = c_1$ and $a_2 = c_2$.

ex: $\mathcal{G} = \{ e^{j2\pi n x}, n \in \mathbb{Z}$ (integers) $\}$ is a basis for $\mathcal{F}$ = real valued periodic continuous functions on $[0,1]$ for field $\mathcal{A} = \mathbb{C}$

pf: We know we can expand any $f \in \mathcal{F}$ in a Fourier series with complex coefficients.

$$f = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n x} .$$