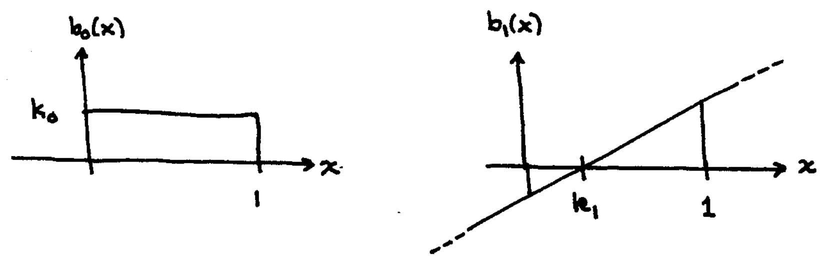
1. Suppose $\{b_n\}_{n=0}^{\infty}$ is an <u>orthonormal basis</u> on domain $D = [0, 1]$. Note that this domain is unit ~~square~~ interval from 0 to 1 rather than $-\frac{1}{2}$ to $\frac{1}{2}$, and b_n is shorthand for $b_n(x)$.

For any continuous real-valued function $f(x)$ on D we have

$$f(x) = \sum_{n=0}^{\infty} a_n b_n(x)$$

where a_n's are suitably chosen real coefficients.

The first two b_n's are plotted below.



a) As shown above, $b_0(x)$ has the constant value k_0. Determine k_0. Explain your answer.

soln: Orthonormal basis $\Rightarrow$ $(b_0, b_0) = 1 = \int_D b_0^2\, dx = \int_0^1 k_0^2\, dx = 1 \cdot k_0^2$

$\therefore \boxed{k_0 = 1}$

b) As shown above, $b_1(x)$ is a straight line crossing the x-axis at k_1. Determine k_1. Explain your answer.

soln: Orthonormal basis $\Rightarrow$ $(b_0, b_1) = 0 = \int_D b_0 b_1\, dx = \int_0^1 1 \cdot b_1\, dx$

$\therefore$ Area under $b_1 = 0$ $\Rightarrow \boxed{k_1 = \frac{1}{2}}$

c) Find a function $f(x)$ such that $a_0 = 0$ and $a_1 = 0$ in its series expansion. Hint: try a polynomial.

soln: Start with $\tilde{f}(x) = x^2$, find a_0, a_1 and let $f(x) = \tilde{f}(x) - a_0 b_0(x) - a_1 b_1(x)$. Then $a_0 = a_1 = 0$ for $f(x)$.

1.c. (cont) For x^2 we have $a_0 = (x^2, b_0) = \int_0^1 x^2 \cdot 1\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

$$a_1 = (x^2, b_1) = \int_0^1 x^2 \cdot k_2(x-\tfrac{1}{2})\,dx = \ldots$$

We need k_2 to continue with this approach.

$$(b_1, b_1) = 1 = \int_0^1 k_2^2 (x-\tfrac{1}{2})^2\,dx = k_2^2\, 2 \cdot \int_0^{1/2} x^2\,dx = 2k_2^2 \frac{x^3}{3}\Big|_0^{1/2}$$

$$1 = 2k_2^2 \frac{1}{24} = \frac{k_2^2}{12} \Rightarrow k_2 = \sqrt{12}$$

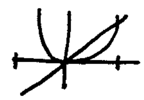
$$a_1 = \int_0^1 x^2 \cdot k_2(x-\tfrac{1}{2})\,dx = k_2 \int_0^1 (x^3 - \tfrac{1}{2}x^2)\,dx$$

$$= \sqrt{12}\left(\frac{x^4}{4} - \frac{1}{2}\frac{x^3}{3}\right)\Big|_0^1 = \sqrt{12}\left(\frac{1}{4} - \frac{1}{6}\right)$$

$$a_1 = \sqrt{12}\,\frac{1}{12} = \frac{1}{\sqrt{12}}$$

$$\therefore f(x) = x^2 - \frac{1}{3} \cdot 1 - \frac{1}{\sqrt{12}}\sqrt{12}(x-\tfrac{1}{2}) = x^2 - x + \frac{1}{6}$$

Easier approach is to start with $(x-\frac{1}{2})^2$ which is $\perp$ b_1 by symmetry and signs:



$$\int_0^1 (x-\tfrac{1}{2})^2 b_1\,dx = 0$$

So $a_1 = 0$. For a_0 we have $a_0 = \int_0^1 (x-\frac{1}{2})^2 \cdot 1\,dx = 2\int_0^{1/2} x^2\,dx$

$$= 2\frac{x^3}{3}\Big|_0^{1/2} = \frac{2 \cdot 1/8}{3} = \frac{1}{12}$$

$$f(x) = (x-\tfrac{1}{2})^2 - \frac{1}{12} = x^2 - x + \frac{1}{4} - \frac{1}{12} = x^2 - x + \frac{1}{6}$$

Either way $\boxed{f(x) = x^2 - x + \frac{1}{6}}$