

3D Graphics Equations

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Abstract—This paper derives a mapping from a point in three dimensions to a point on a two-dimensional image plane.

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I. INTRODUCTION

The MATLAB function `Two_Dim_View.m` projects points in three dimensions onto a two-dimensional image plane. Starting with a right hand three-dimensional frame, the text here shows the derivation of the equations for the projection.

To specify the image plane, one must specify a point, s_0 (called `Focal_point` in MATLAB), in the 3-dimensional space that is straight ahead and will, therefore, appear in the center of the image plane. The image plane is also assumed to pass through the focal point and have the focal point as its origin. The idea behind the derivation is to draw lines from a viewpoint, (called `view_point` in MATLAB), to the points in three-dimensions and determine where those lines intersect the image plane. The image plane is perpendicular to the line of sight connecting the viewpoint and s_0 .

Fig. 1 shows the geometry of the imaging problem after the focal point, s_0 , is subtracted from every point in the 3-dimensional space, which is equivalent to moving the origin of the 3-dimensional space to s_0 . **All values from here onward are in this new frame of reference.**

The viewpoint is now at v_0 (called `v` in MATLAB). The distance, d (called `d` in MATLAB), from viewpoint v_0 to the image plane is equal to the distance from the viewpoint to the focal point and determines how far projected points will be from the origin of the image plane.

Fig. 1 shows a point p_0 in the 3-dimensional space that gets projected onto point p_0 on the image plane. Axes for the 3-dimensional space are shown in black, and axes for the image plane are shown in orange. Fig. 2 shows the same situation as Fig. 1 but from a top view. Note that, because the image plane is tilted relative to the x,y -plane, p_0 is offset from the v_{10} axis of the image plane in the top view.

II. DEFINITIONS OF TERMS

$\vec{s}_0 \equiv$ focal point

$\equiv$ vector subtracted from every point in 3-D space to move origin, (0, 0, 0), of 3-D space to center of image plane

$v_0 \equiv$ viewpoint – focal point

$\theta \equiv$ direction (or heading) to viewpoint in x,y -plane (math direction) measured counter-clockwise from $+x$ -axis

$\equiv \angle +x\text{-axis to } -\vec{v}_0 \text{ in counter-clockwise direction}$

$\phi \equiv$ pitch ($\angle$ between $\vec{v}_0$ and z -axis)

$v_{0xy} \equiv v_0$ projected onto x,y -plane

$d \equiv$ distance from v_0 to origin $= |\vec{v}_0|$

$\mathbf{v}$ $\equiv$ 2-dim image plane $\perp$ $\mathbf{v}_0$ (or $\perp$ line from $\mathbf{v}_0$ to (0, 0, 0)) passing thru origin

(Note: $\mathfrak{J}$ is symbol font capital J)

$\mathbf{v}_{10}$ $\equiv$ unit vector in image plane $\perp$ $\mathbf{v}_0$ and lying in x,y -plane $= (v_{10x}, v_{10y}, v_{10z})$

$\mathbf{v}_{20}$ $\equiv$ unit vector in image plane $\perp$ $\mathbf{v}_{10}$ and $\perp$ $\mathbf{v}_0$ (or $\mathbf{v}_{30}$) $= (v_{20x}, v_{20y}, v_{20z})$

$\mathbf{v}_{30}$ $\equiv$ unit vector out of image plane in direction of $\mathbf{v}_0 = (v_{30x}, v_{30y}, v_{30z})$

$\mathbf{v}$ $\equiv$ plane defined by span of $\mathbf{v}_{10}$ and $\mathbf{v}_{20}$

$=$ plane consisting of vectors $\perp$ to $\mathbf{v}_{30}$ since $\mathbf{v}_{30}$ is normal to plane $\mathfrak{J}$

$\mathbf{p}_0 \equiv (p_x, p_y, p_z) \equiv$ point in 3-D to be projected on image plane

$\mathbf{p}_0 \equiv (\rho_x, \rho_y, \rho_z) \equiv$ location of projection of $\mathbf{p}_0$ on image plane in coords $\mathbf{v}_{10}, \mathbf{v}_{20}$, and $\mathbf{v}_{30}$

II. DERIVED EQUATIONS

Angle θ is determined from the coordinates of $\mathbf{v}_0$:

$$\theta = \tan^{-1} \left(\frac{v_{0y}}{v_{0x}} \right).$$

Note that the MATLAB code employs the two-argument inverse tangent function in order to avoid an error of π radians or 180° .

$$d = |\mathbf{v}_0|$$

$$\phi = \cos^{-1} \left(\frac{v_{0z}}{d} \right).$$

Trigonometry gives the distance in the x,y -plane from the origin to the viewpoint:

$$d_{xy} = d \sin \phi.$$

$$\mathbf{v}_{30} = \frac{\mathbf{v}_0}{|\mathbf{v}_0|} = \frac{\mathbf{v}_0}{d}$$

$$\mathbf{v}_0 = (d_{xy} \cos \theta, d_{xy} \sin \theta, d \sin \phi) = d(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$$

$$\mathbf{v}_{30} = \frac{\mathbf{v}_0}{d} = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \text{ lies in the } \mathbf{v}_0 \text{ direction}$$

$$\angle \mathbf{v}_{10} = \theta' + 90 = 180^\circ - \theta \text{ in } xy\text{-plane}$$

$$v_{10x} = \cos(\theta + 90^\circ) \quad v_{10y} = \sin(\theta + 90^\circ) \quad v_{10z} = 0$$

$$|\mathbf{v}_{10}| = 1$$

$$\mathbf{v}_{10} = (\cos(\theta + 90^\circ), \sin(\theta + 90^\circ), 0)$$

$$\mathbf{v}_{10} = (-\sin\theta, \cos\theta, 0)$$

$$\mathbf{v}_{20} \perp \mathbf{v}_{30}$$

$$\mathbf{v}_{20} = (-\cos\theta\cos\phi, -\sin\theta\cos\phi, \sin\phi)$$

$$\mathbf{v}_{20} \circ \mathbf{v}_{30} = (-\cos\theta\cos\phi, -\sin\theta\cos\phi, \sin\phi) \circ (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

$$\mathbf{v}_{20} \circ \mathbf{v}_{30} = -\cos^2\theta\cos\phi\sin\phi - \sin^2\theta\cos\phi\sin\phi + \cos\phi\sin\phi$$

$$\mathbf{v}_{20} \circ \mathbf{v}_{30} = -\cos\phi\sin\phi + \cos\phi\sin\phi = 0$$

III. DERIVATION: PROJECTION OF POINT ONTO IMAGE PLANE

Equation for line l from $\mathbf{v}_0$ to $\mathbf{p}_0$ and image plane, $\mathfrak{I}$, is

$$l = \mathbf{v}_0 + \lambda(\mathbf{p}_0 - \mathbf{v}_0) \text{ for real scalar } \lambda.$$

The image point, $\mathbf{p}_0$, lies on line l and in image plane, $\mathfrak{I}$, which is normal to $\mathbf{v}_{30}$. Thus,

$$(\mathbf{p}_0 \equiv \mathbf{v}_0 + \lambda_0(\mathbf{p}_0 - \mathbf{v}_0)) \circ \mathbf{v}_{30} = 0$$

$$\mathbf{v}_0 \circ \mathbf{v}_{30} + \lambda_0(\mathbf{p}_0 - \mathbf{v}_0) \circ \mathbf{v}_{30} = 0$$

$$\lambda_0 = -\frac{\mathbf{v}_0 \circ \mathbf{v}_{30}}{(\mathbf{p}_0 - \mathbf{v}_0) \circ \mathbf{v}_{30}} = -\frac{d\mathbf{v}_{30} \circ \mathbf{v}_{30}}{(\mathbf{p}_0 - d\mathbf{v}_{30}) \circ \mathbf{v}_{30}} = -\frac{d}{\mathbf{p}_0 \circ \mathbf{v}_{30} - d}$$

Now substitute λ_0 back into the formula for $\mathbf{p}_0$.

$$\mathbf{p}_0 = \mathbf{v}_0 + \lambda_0(\mathbf{p}_0 - \mathbf{v}_0) = \mathbf{v}_0 - \frac{d}{\mathbf{p}_0 \circ \mathbf{v}_{30} - d}(\mathbf{p}_0 - \mathbf{v}_0)$$

$$\mathbf{p}_0 = \mathbf{v}_0 \frac{\mathbf{p}_0 \circ \mathbf{v}_{30} - d}{\mathbf{p}_0 \circ \mathbf{v}_{30} - d} - \frac{d}{\mathbf{p}_0 \circ \mathbf{v}_{30} - d}(\mathbf{p}_0 - \mathbf{v}_0)$$

$$\mathbf{p}_0 = \frac{\mathbf{v}_0(\mathbf{p}_0 \circ \mathbf{v}_{30} - \cancel{d} + \cancel{d}) - d\mathbf{p}_0}{\mathbf{p}_0 \circ \mathbf{v}_{30} - d}$$

Replace $\mathbf{v}_0$ with $d\mathbf{v}_{30}$ for a more numerically stable calculation.

$$\mathbf{p}_0 = \frac{d\mathbf{v}_{30}(\mathbf{p}_0 \circ \mathbf{v}_{30}) - d\mathbf{p}_0}{\mathbf{p}_0 \circ \mathbf{v}_{30} - d}$$

$$\mathbf{p}_0 = d \frac{\mathbf{v}_{30}(\mathbf{p}_0 \circ \mathbf{v}_{30}) - \mathbf{p}_0}{\mathbf{p}_0 \circ \mathbf{v}_{30} - d}.$$

IV. MATLAB CODE

A.