

TOOL: Partial fraction expansion forms for the inverse Laplace transform depend on the type(s) of root(s) term(s) found in the denominator. The values of the denominator roots are referred to as poles.

We assume $X(s)$ is expressed as a polynomial over a polynomial with the coefficient of the highest power of s in the denominator being 1.

NOTE: $X(s)$ may have any combination of the types of partial fraction terms shown below.

Single Real Root (or Pole)

$$X(s) = \frac{N(s)}{(s+a)D(s)} = \frac{A}{s+a} + \text{other partial fraction terms}$$

where $N(s)$ and $D(s)$ are the numerator and remaining denominator polynomials in s that do *not* have $(s+a)$ as a root term.

In the time domain, we get $Ae^{-at}u(t)$ for the single real root.

Repeated Real Root (or Pole)

$$X(s) = \frac{N(s)}{(s+a)^n D(s)} = \frac{A_n}{(s+a)^n} + \frac{A_{n-1}}{(s+a)^{n-1}} + \dots + \frac{A_1}{s+a} + \text{other terms}$$

where $N(s)$ and $D(s)$ are the numerator and remaining denominator polynomials in s that do *not* have $(s+a)$ as a root term.

Note that we must have every power of the root in the denominator, but some of the A coefficients may be found to be zero, causing those terms to disappear.

In the time domain, we get $A_n \frac{t^n e^{-at}}{n!} u(t)$ for $\frac{A_n}{(s+a)^n}$.

Complex Conjugate Roots (or Poles)

Complex roots for polynomials with real coefficients always come in conjugate pairs.

$$X(s) = \frac{N(s)}{(s+a+j\omega)(s+a-j\omega)D(s)} = \frac{A}{s+a+j\omega} + \frac{A^*}{s+a-j\omega} + \text{other terms}$$

Or we use a formulation that avoids complex numbers and does both roots at once:

$$X(s) = \frac{N(s)}{[(s+a)^2 + \omega^2]D(s)} = \frac{As+B}{(s+a)^2 + \omega^2} + \text{other terms}$$

Note: $s^2 + bs + c$ has complex roots if $\left(\frac{b}{2}\right)^2 < c$.

$$\text{In this case, } a = \frac{b}{2} \text{ and } \omega = \sqrt{c - \left(\frac{b}{2}\right)^2}.$$

Complex roots correspond to sinusoids in the time domain. If the root has a real part, the sinusoid is growing or decaying with an exponential envelope. The rate of growth or decay is dictated by the real part of the pole value.

In the time domain, we get $[Ae^{-at} \cos(\omega t) + \frac{B-aA}{\omega} e^{-at} \sin(\omega t)]u(t)$ for $\frac{As+B}{(s+a)^2 + \omega^2}$.