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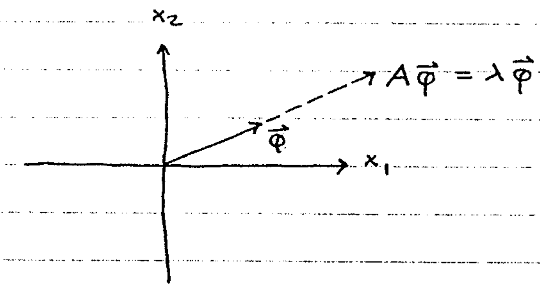
Linear Algebra - Invariants - Eigenvalues, Eigenvectors
5 Apr 1991



Tool: An eigenvector $\vec{\varphi}$ of matrix A is a vector whose direction remains the same when it is multiplied by A:

$$A\vec{\varphi} = \lambda\vec{\varphi}$$

where λ is a scalar "eigenvalue".

Note: λ can be positive or negative or complex.

ex: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \vec{\varphi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \lambda_1 = -1$

$\vec{\varphi}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \lambda_2 = 1$

$$A\vec{\varphi}_1 = \begin{bmatrix} -1\cdot 1 + 0\cdot 0 \\ 0\cdot 1 + 1\cdot 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} (-1)\,1 \\ (-1)\,0 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1\vec{\varphi}_1$$

$$A\vec{\varphi}_2 = \begin{bmatrix} -1\cdot 0 + 0\cdot 1 \\ 0\cdot 0 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (1)\,0 \\ (1)\,1 \end{bmatrix} = (1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda_2\vec{\varphi}_2$$

ex: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad \vec{\varphi}_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}$ $\quad \lambda_1 = j$

$\vec{\varphi}_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix}$ $\quad \lambda_2 = -j$

Note: An nxn matrix has n eigenvalues (but some may be zero).

Neil E. Cotter   Linear Algebra – Invariants – Eigenvalues, Eigenvectors (cont.)
5 Apr 1991

Derivation: $A\vec{\varphi} = \lambda\vec{\varphi}$ (eq'n satisfied by eigenvector)

If this eq'n is satisfied, then

$$A\vec{\varphi} - \lambda\vec{\varphi} = \vec{0}.$$

We can multiply $\vec{\varphi}$ by the identity matrix $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$:

$$A\vec{\varphi} - \lambda I\vec{\varphi} = \vec{0}.$$

Distributivity applies to matrix multiplication:

$$(A - \lambda I)\vec{\varphi} = \vec{0}$$

We want to find a nonzero (nontrivial) vector $\vec{\varphi}$ satisfying this eq'n. Since the right-hand side is zero, we can have nonzero φ only if

$A - \lambda I$ is <u>singular</u>.

This is equivalent to saying the determinant is zero:

$$\det(A - \lambda I) = 0.$$

The eigenvalues λ are the roots of this eq'n.

To find eigenvectors, substitute the value of λ into $A - \lambda I$ and find the φ satisfying

$$(A - \lambda I)\vec{\varphi} = 0.$$

For a 2x2 matrix we can choose $\vec{\varphi} = \begin{bmatrix} 1 \\ x \end{bmatrix}$ and solve for x.

ex: $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1 \\ 3 & 5-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(5-\lambda) - 3(-1)$$

$$= \lambda^2 - 6\lambda + 5 + 3$$

$$= \lambda^2 - 6\lambda + 8 = 0$$

$$\lambda = -\frac{-6}{2} \pm \sqrt{\left(\frac{-6}{2}\right)^2 - 8} = 4 \text{ or } 2$$

eigenvalues: $\lambda_1 = 4$ $\lambda_2 = 2$

eigenvector $\vec{\varphi}_1$: $(A - \lambda_1 I)\,\vec{\varphi}_1 = \vec{0}$

$$\begin{bmatrix} 1-4 & -1 \\ 3 & 5-4 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

singular matrix because 2nd row = -1 · first row

$x = -3$ $\vec{\varphi}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Neil E. Cotter Linear Algebra - Invariants - Eigenvalues, Eigenvectors (cont.)
5 Apr 1991

eigenvector $\vec{\varphi}_2$: $(A - \lambda_2 I)\vec{\varphi}_2 = \vec{0}$

$$\begin{bmatrix} 1-2 & -1 \\ 3 & 5-2 \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = -1 \qquad \vec{\varphi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

check: $A\vec{\varphi}_1 \stackrel{?}{=} \lambda_1 \vec{\varphi}_1$ $\qquad A\vec{\varphi}_2 \stackrel{?}{=} \lambda_2 \vec{\varphi}_2$

$$A\vec{\varphi}_1 = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + (-1)(-3) \\ 3\cdot 1 + 5(-3) \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix} = 4\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \lambda_1 \vec{\varphi}_1 \checkmark$$

$$A\vec{\varphi}_2 = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + (-1)(-1) \\ 3\cdot 1 + 5(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda_2 \vec{\varphi}_2 \checkmark$$

Linear Algebra – Invariants –
Eigenvalues, Eigenvectors (cont.)

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$$A\vec{\varphi} = \lambda\vec{\varphi} \qquad \lambda \equiv \text{eigenvalue}$$
$$\underset{\text{matrix}}{\uparrow} \quad \underset{\text{scalar}}{\uparrow} \qquad \vec{\varphi} \equiv \text{eigenvector}$$

• Problem is to find λ, φ given A

$A\vec{\varphi} - \lambda\vec{\varphi} = \vec{0}$ • soln iff $\overset{A-\lambda I}{\cancel{A\vec{\varphi}-\lambda\vec{\varphi}}}$ is singular

Then $\det(\overset{A-\lambda I}{\cancel{A\vec{\varphi}-\lambda\vec{\varphi}}}) \equiv \left|\overset{A-\lambda I}{\cancel{A\vec{\varphi}-\lambda\vec{\varphi}}}\right| = 0$

• Note that $\lambda I\vec{\varphi} = \lambda\vec{\varphi}$.

ex: $A = \begin{bmatrix} 5 & 3 \\ 3 & 13 \end{bmatrix}$ $\det(A-\lambda I) = \det\begin{bmatrix} 5-\lambda & 3 \\ 3 & 13-\lambda \end{bmatrix} = 0$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

$\|$

$(5-\lambda)(13-\lambda) - 3\cdot 3 = 0$

$\lambda_1 = 4, \ \lambda_2 = 14$ • two eigvals

• $(4-\lambda)(\overset{14}{\cancel{14}}-\lambda) = \lambda^2 - 18\lambda + 56$

plug in λ_1, find $\vec{\varphi}_1$:

$$\begin{bmatrix} 5-4 & 3 \\ 3 & 13-4 \end{bmatrix} \vec{\varphi}_1 = 0$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 0 \qquad \varphi_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Find $\vec{\varphi}_2$:

$$\begin{bmatrix} 5-14 & 3 \\ 3 & 13-14 \end{bmatrix} \vec{\varphi}_2 = 0$$

$$\begin{bmatrix} -9 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0 \qquad \varphi_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$