EX: Calculate the odds (or probabilities) of the following 5-card poker hands:

a) royal flush

b) four-of-a-kind

c) straight-flush (excluding royal flush)

d) full house

e) flush (excluding straight-flush)

f) straight (excluding straight-flush)

g) three-of-a-kind

h) two-pair

i) one-pair

j) high card

Also, verify that the probabilities sum to unity. Assume a 52-card deck. (The number of possible 5-card hands is ${}_{52}C_5$ = 2,598,960.)

SOL'N: a) A royal flush is ace, king, queen, jack, and ten of the same suit.

If we order the 5-card hand from highest card to lowest, the first card will be an ace. There are four possible suits for the ace. After that, the other four cards are completely determined. Thus, there are 4 possible royal flushes:

$$\text{\# royal flushes} = {}_4C_1 = 4$$

Dividing by the number of possible hands gives the probability:

$$P(\text{royal flush}) = \frac{4}{2{,}598{,}960} = 1.539 \cdot 10^{-6} \text{ or 1 in 649,740}$$

b) A straight-flush (excluding royal flush) is all cards the same suit and showing consecutive numbers (but not the highest 5 consecutive numbers).

If we order the 5-card hand from highest number to lowest, the first card may be one of the following: king, queen, jack, 10, 9, 8, 7, 6, or 5. (Note: the ace may be the card above a king or below a 2, but we would have a royal flush if it were the card above the king.) There are 9 possibilities. After the first card, whose suit we may choose in 4 ways, the remaining cards are completely determined.

$$\#\text{ straight flushes} = 9\ {}_4C_1 = 9 \cdot 4 = 36$$

Subtracting the number of royal flushes and dividing by the number of possible hands gives the probability:

$$P(\text{straight - flush}) = \frac{36}{2{,}598{,}960} = 1.385 \cdot 10^{-5} \text{ or 1 in } 72{,}193\tfrac{1}{3}.$$

c) A four-of-a-kind is four cards showing the same number plus any other card.

If we order the 5-card hand with the four-of-a-kind first, we have ${}_{13}C_1$ choices for the number showing on the first four cards. Since we will have all four suits, we have only ${}_4C_4 = 1$ way to choose the suits. The remaining card will be any of the 48 remaining cards:

$$\#\text{ 4-of-a-kinds} = {}_{13}C_1 \cdot {}_4C_4 \cdot {}_{48}C_1 = 13 \cdot 1 \cdot 48 = 624$$

Dividing by the number of possible hands gives the probability:

$$P(\text{4 - of - a - kind}) = \frac{624}{2{,}598{,}960} = 2.401 \cdot 10^{-4} \text{ or 1 in 4165}$$

d) A full house is three cards showing the same number plus a pair.

If we order the 5-card hand with the three-of-a-kind first, we have ${}_{13}C_1$ choices for the number showing on the first three cards. For the choice of suits, we have three out of four possible suits or ${}_4C_3 = 4$ possibilities. For the remaining pair, we have ${}_{12}C_1$ choices for the number showing on the two cards. For the choice of suits, we have two out of four possible suits or ${}_4C_2 = 6$ possibilities.

$$\text{\# full houses} = {}_{13}C_1 \cdot {}_4C_3 \cdot {}_{12}C_1 \cdot {}_4C_2 = 13 \cdot 4 \cdot 12 \cdot 6 = 3744$$

Dividing by the number of possible hands gives the probability:

$$P(\text{full house}) = \frac{3744}{2{,}598{,}960} = 1.441 \cdot 10^{-3} \text{ or 1 in } 694\tfrac{1}{6}$$

e) A flush (excluding straight-flush) is all cards the same suit (but not a straight).

If we pick the suit first, we have ${}_4C_1 = 4$ choices. For that suit, there are 13 cards from which we choose 5. Thus, we have ${}_{13}C_5 = 1287$ choices. Subtracting the straights, (which may start with ace, king, queen, ... 5), reduces this number by 10.

$$\text{\# flushes (not straight)} = {}_4C_1 \cdot ({}_{13}C_5 - 10) = 4 \cdot 1277 = 5108$$

Dividing by the number of possible hands gives the probability:

$$P(\text{flush not straight}) = \frac{5108}{2{,}598{,}960} = 1.965 \cdot 10^{-3} \text{ or 1 in } 508.8$$

f) A straight (excluding straight-flush) is five cards showing consecutive numbers (but not all of the same suit).

If we order the 5-card hand from highest number to lowest, the first card may be one of the following: ace, king, queen, jack, 10, 9, 8, 7, 6, or 5. (Note: the ace may be the card above a king or below a 2.) There are 10 possibilities. After the first card, the numbers showing on the remaining four cards are completely determine. If we allow flushes, including royal flushes, there are four possible suits for each of the cards. (Note that this holds because the cards all show different numbers, and there are four suits for each number.)

$$\text{\# straights} = 10 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 = 10 \cdot 4^5 = 10{,}240$$

Subtracting the number of straight flushes and royal flushes and dividing by the number of possible hands gives the probability:

$$P(\text{straight not flush}) = \frac{10{,}240 - 36 - 4)}{2{,}598{,}960} = 3.925 \cdot 10^{-3} \text{ or 1 in } 254.8$$

g) A three-of-a-kind is three cards showing the same number plus two cards that do not form a pair or create a four-of-a-kind.

If we order the 5-card hand with the three-of-a-kind first, we have ${}_{13}C_1$ choices for the number showing on the first three cards. Since we will have three out of four suits, we have ${}_4C_3 = 4$ ways to choose the suits. The remaining two cards must show different numbers than the four-of-a-kind and each other. There are ${}_{12}C_2$ choices for these numbers. The last two cards may have any of the four suits, however.

$$\#\text{ 3-of-a-kind} = {}_{13}C_1 \cdot {}_4C_3 \cdot {}_{12}C_2 \cdot {}_4C_1 \cdot {}_4C_1 = 13 \cdot 4 \cdot 66 \cdot 4 \cdot 4 = 54{,}912$$

Dividing by the number of possible hands gives the probability:

$$P(\text{3-of-a-kind}) = \frac{54{,}912}{2{,}598{,}960} = 2.113 \cdot 10^{-2} \text{ or 1 in 47.3.}$$

h) Two-pairs is two cards showing the same numbers and another two cards showing the same numbers (but not all four numbers the same) plus one extra card (not the same as any of the other numbers).

If we order the 5-card hand with the two pairs first, we have ${}_{13}C_2$ choices for the two numbers showing on the two pairs. Each pair will have two out of four suits. Thus, we have ${}_4C_2 \cdot {}_4C_2 = 6 \cdot 6 = 36$ ways to choose the suits. The remaining card must show a different number than the two pairs. There are ${}_{11}C_1$ choices for this number. The last card may have any of four suits.

$$\#\text{ 2-pair} = {}_{13}C_2 \cdot {}_4C_2 \cdot {}_4C_2 \cdot {}_{11}C_1 \cdot {}_4C_1 = 78 \cdot 6 \cdot 6 \cdot 11 \cdot 4 = 123{,}552$$

Dividing by the number of possible hands gives the probability:

$$P(\text{2-pair}) = \frac{123{,}552}{2{,}598{,}960} = 4.754 \cdot 10^{-2} \text{ or 1 in 21.}$$

i) One-pair is two cards showing the same numbers and another three cards all showing different numbers.

If we order the 5-card hand with the pair first, we have ${}_{13}C_1$ choices for the number showing on the pair. The pair will have two out of four suits. Thus, we have ${}_4C_2 = 6$ ways to choose the suit. The remaining three cards must show different numbers than the pair and each other. There are ${}_{12}C_3$ choices for these numbers. The last three cards may each have any of four suits.

$$\# \text{ 1-pair} = {}_{13}C_1 \cdot {}_4C_2 \cdot {}_{12}C_3 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 = 13 \cdot 6 \cdot 220 \cdot 4 \cdot 4 \cdot 4 = 1{,}098{,}240$$

Dividing by the number of possible hands gives the probability:

$$P(\text{1 - pair}) = \frac{1{,}098{,}240}{2{,}598{,}960} = 4.226 \cdot 10^{-1} \text{ or 1 in 2.37.}$$

j) High card means we must avoid higher-ranking hands. All higher-ranked hands include a pair, a straight, or a flush.

We now count the number of possible high-card hands

Because the numbers showing on the cards must be five different numbers, we have ${}_{13}C_5$ choices for the five numbers showing on the cards. Each of the cards may have any of four suits.

$${}_{13}C_5 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 = 1287 \cdot 4^5 = 1{,}317{,}888$$

We subtract the number of straights, flushes, and royal flushes. (Note that we avoided having any pairs or more of a kind.) Dividing the difference just calculated by the number of possible hands gives the probability:

$$P(\text{high - card}) = \frac{1{,}317{,}888 - 10{,}200 - 5{,}108 - 36 - 4}{2{,}598{,}960} = \frac{1{,}302{,}540}{2{,}598{,}960}$$

or

$$P(\text{high - card}) = 0.501 \text{ or 1 in 2.}$$

REF: *Probability: 5-card Poker Hands*, Tom Ramsey, http://www.math.hawaii.edu/~ramsey/Probability/PokerHands.html, Feb. 1, 2008.