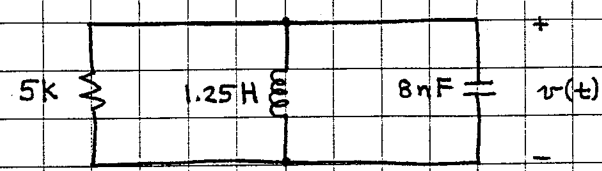
Ref: Nilsson & Riedel, Rev 6, Prob 8.10



a) Calculate roots of characteristic eq'n for $v(t)$.
b) Is response over–, under–, or critically damped?
c) What R yields damped freq of 6 k rad/s?
d) Find roots of characteristic eq'n for R found in (c).
e) Find R for critically damped response.

ans: a) $s_1 = -5\text{k}/\text{s}$, $s_2 = -20\text{k}/\text{s}$ (units are actually k rad/s)
b) overdamped
c) $R = 7.8125 \text{ k}\Omega$
d) $s_1 = -8\text{k} + j6\text{k} /\text{s}$, $s_2 = -8\text{k} - j6\text{k} /\text{s}$ (actually k rad/s)
e) $R = 6.25 \text{ k}\Omega$

soln: a) From Kirchhoffs law, we know the total current flowing out of the top center node is zero:

$$i_R + i_L + i_C = 0\text{A}$$

We calculate each of the currents:

$i_R = \dfrac{v}{R}$ $\quad$ $i_L = ?$ We have to start with $v = L\dfrac{di_L}{dt}$ and solve for i_L.

Multiply both sides of $v = L\dfrac{di_L}{dt}$ by $\dfrac{1}{L}dt$ and integrate:

$$\frac{1}{L}\int_{t'=0}^{t'=t} v\, dt' = \int_{i_L(t=0)}^{i_L(t)} di_L$$

We use t' as the dummy var of integration to avoid confusion with time t. We also match the limits of integration to the type of differential: t for dt, i_L for di_L. Furthermore, the lower and upper limits must be evaluated at the same points in time: $t'=0$ and $i_L(t=0)$, and $t'=t$ and $i_L(t)$.

The righthand side of the preceding eq'n simplifies:

$$\frac{1}{L}\int_{t'=0}^{t'=t} v\,dt' = \int_{i_L(t=0)}^{i_L(t)} di_L = i_L(t) - i_L(t=0)$$

Solving for $i_L(t)$ by moving the $i_L(t=0)$ to the other side gives:

$$i_L(t) = \frac{1}{L}\int_{t'=0}^{t'=t} v\,dt' + i_L(t=0)$$

Finally, for i_C we use $i_C = C\dfrac{dv}{dt}$.

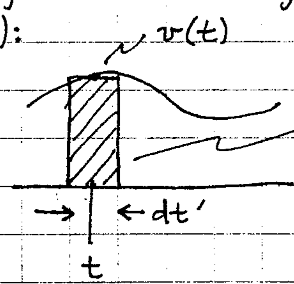
For $i_R + i_L + i_C = 0$ we have:

$$\frac{v}{R} + \frac{1}{L}\int_{t'=0}^{t'=t} v\,dt' + i_L(t=0) + C\frac{dv}{dt} = 0\text{ A}$$

To eliminate the integral, we differentiate both sides. (This also gets rid of $i_L(t=0)$ which is a constant.)

$$\frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2} = 0\text{ A/s} \quad \text{(at time } t\text{)}$$

Note that $\dfrac{d}{dt}\displaystyle\int^{t'=t} v\,dt' = v(t)$ because the rate of change of the integral of v — at time t — is just $v(t)$:



change in $\int v\,dt'$ in time dt' at absolute time t is $\lim_{dt'\to 0} v(t)\cdot dt'$.

The rate of change of $\int v\,dt'$ at time t is

$$\lim_{dt'\to 0}\frac{1}{dt'}v(t)\cdot dt' = \lim_{dt'\to 0} v(t) = v(t).$$

Now try sol'n $v(t) = Ae^{st}$ in diff. eq'n:

$$\frac{dv}{dt} = Ase^{st} \qquad \frac{d^2v}{dt} = As^2e^{st}$$

$$\frac{1}{R}Ase^{st} + \frac{1}{L}Ae^{st} + CAs^2e^{st} = 0 \text{ A/s}$$

Note: A is some constant we must determine, and
s is another " " " " .

We determine s from the characteristic eq'n we obtain by factoring Ae^{st} out of the above eq'n:

$$\left(\frac{1}{R}s + \frac{1}{L} + Cs^2\right)Ae^{st} = 0 \quad \text{A/s}$$

Clearly, either $Ae^{st} = 0$ or $\frac{1}{R}s + \frac{1}{L} + Cs^2 = 0$.

If $Ae^{st} = 0$ then $v(t) = 0$ always. This is impossible since we can build an RLC and observe that $v(t) \neq 0$ V.

We conclude that, (if Ae^{st} does indeed work as a sol'n), $\frac{1}{R}s + \frac{1}{L} + Cs^2 = 0$. This is the characteristic eq'n.

This is a quadratic eq'n that we put in a convenient form by dividing thru by C:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

We also define a convenient notation:

$$s^2 + 2\alpha s + \omega_0^2 = 0 \qquad \alpha \equiv \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

We solve for s and discover there are two sol'ns:

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

case I: If $\alpha > \omega_0$ we get s_1, s_2 real and our $v(t)$ will consist of the sum of two decaying exponentials. (Note that $s_1, s_2 < 0$ so $e^{s_1 t}$ and $e^{s_2 t}$ decay.)

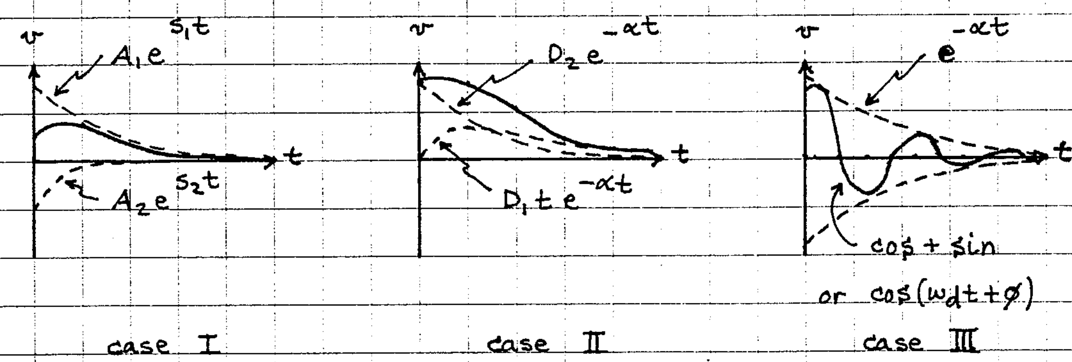
$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ overdamped

case II: If $\alpha = \omega_0$ we get $s_1 = s_2 = \alpha$ real and our $v(t)$ will consist of the sum of a decaying exponential and t times that same decaying exponential. (Note that we have not explained *why* this happens, but we get this result if we take the limit of the case I soln as $s_1 - s_2 \to 0$.)

$v(t) = (D_1 t + D_2) e^{-\alpha t}$ critically damped

case III: If $\alpha < \omega_0$ we get $s_1 = s_2^*$ complex and conjugate. Our $v(t)$ will consist of sinusoid (at angular frequency $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$) multiplied by an exponentially decaying envelope $e^{-\alpha t}$.

$v(t) = (B_1 \cos \omega_d t + B_2 \sin \omega_d t) e^{-\alpha t}$



Having found s_1 and s_2, we find A_1 and A_2 (or D_1, D_2, or B_1, B_2) by making our $v(t)$ satisfy initial conditions (i.e. at $t = 0^+$): $v(t=0^+)$ and $\frac{dv}{dt}(t=0^+)$. (See later problems.)

- We are now ready to answer part (a):

char eqn roots $s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \cdot 5\text{k}\Omega \cdot 8\text{nF}} = \frac{1}{80\mu} \text{ rad/s} = 12.5 \text{ k rad/s}$$

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{1.25\text{H}\ 8\text{nF}} = \frac{1}{10\text{n}}\ \text{rad/s} = 100\text{M}\ \text{rad}^2/\text{s}^2 \text{ or } (10\text{k rad/s})^2$$

$$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{(12.5\text{k})^2 - (10\text{k})^2}\ \text{rad/s}$$

$$= 2.5\text{k}\sqrt{5^2 - 4^2}\ \text{rad/s}$$

$$= 2.5\text{k} \cdot 3 = 7.5\text{k}\ \text{rad/s}$$

$$s_1, s_2 = -12.5\text{k} \pm 7.5\text{k}\ \text{rad/s}$$

$$s_1 = -5\text{k rad/s}, \quad s_2 = -20\text{k rad/s}$$

b) $\alpha > \omega_0$ (real roots $s_1 \neq s_2$) so is overdamped.

c) $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 6\text{ k rad/s}$ desired

$\omega_0^2 = \frac{1}{LC}$ (unaffected by R) is still $(10\text{ k})^2\ \text{rad}^2/\text{s}^2$

$\omega_d^2 = \omega_0^2 - \alpha^2$ from squaring both sides of $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

or $\alpha^2 = \omega_0^2 - \omega_d^2$ or $\alpha = \sqrt{\omega_0^2 - \omega_d^2} = \sqrt{(10\text{k})^2 - (6\text{k})^2}\ \text{rad/s}$

or $\alpha = 8\text{k rad/s}$

Now $\alpha = \frac{1}{2RC} \Rightarrow R = \frac{1}{2\alpha C} = \frac{1}{2 \cdot 8\text{k} \cdot 8\text{n}}\ \text{rad/s} = \frac{1\text{M}}{128}\ \text{rad/s}$

$R = 7.8125\ \text{k}\Omega$

d) $s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8\text{k} \pm j6\text{k rad/s}$ (since $\omega_d = 6\text{k rad/s}$)

e) Critically damped when $s_1 = s_2 \Rightarrow \alpha = \omega_0 = 10\text{k rad/s}$

$R = \frac{1}{2\alpha C} = \frac{1}{2 \cdot 10\text{k} \cdot 8\text{n}}\ \text{rad/s} = \frac{1\text{M}}{160} = 6.25\ \text{k}\Omega$