May 1990 Real Analysis - Measure — Definition, Measurable sets and functions

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measure of set $S \equiv mS \equiv$ min total length of open intervals covering set S

ex: $S = (0, 1/4) \cup (3/4, 1)$ is covered by intervals $(0, 1/4)$ and $(3/4, 1)$ whose total length is $1/4 + 1/4 = 1/2$. We also note that no other collection of open intervals whose union contains (covers) S will have combined length $< 1/2$.

$\therefore\ mS = 1/2$

set S is *measurable* $\equiv$ For every set X that one might choose,

$$mX = m(X \cap S) + m(X \cap \sim S)$$

where $\sim S$ is complement of S

ex: $mS = 0$ implies S is *measurable*

pf: we always have $mX \le m(X \cap S) + m(X \cap \sim S)$. This holds because we have split X into two pieces, and the coverings for the two pieces may be less efficient than the covering for X.

Now $X \cap S \subset S$ so $m(X \cap S) \le m(S) = 0$ since we can use the same covering for $X \cap S$ that we used for S and have zero total length for this covering. Since $m(\,) \ge 0$ always, we conclude that $m(X \cap S) = 0$.

Also $X \cap \sim S \subset X$ so $m(X \cap \sim S) \le mX$.

$$\therefore\ mX \ge m(X \cap \sim S) + m(X \cap S)$$

Since we also had $mX \le m(X \cap \sim S) + m(X \cap S)$ we conclude that $mX = m(X \cap \sim S) + m(X \cap S)$. Then by definition S is *measurable*.

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The family $\mathcal{M}$ of measurable sets contains:

1) $\emptyset$ empty set
2) $\mathbb{R}$ entire real numbers
3) (a,b) any open interval or even any open set
4) $[a,b]$ any closed interval " " " closed "
5) $\sim S$ complement of S for any S measurable
6) $\bigcup_i S_i$ union of countably infinite collection of measurable sets S_i
7) $\bigcap_i S_i$ intersection " " " " " " " "
8) (a,∞) half-line

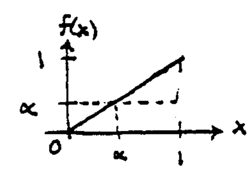
$\mathcal{M}$ is a σ-algebra or Borel field $\equiv$ Items (5)-(7) above mean that $\mathcal{M}$ is a σ-algebra (another name is Borel field). The idea is that $\mathcal{M}$ is closed under the operations of union and intersection: $m_1, m_2 \in \mathcal{M}$ then $m_1 \cap m_2 \in \mathcal{M}$ and $m_1 \cup m_2 \in \mathcal{M}$.

$\mathcal{B}$ is the collection of Borel sets $\equiv$ Smallest sigma algebra containing all open sets.

ex: Every Borel set is measurable.

$f(x)$ is a measurable function $\equiv$ For any number α = real # or $\pm\infty$ $\{x : f(x) > \alpha\}$ is measurable

ex: $f(x) = x$ on $D = [0,1]$



$m\{x : f(x) > \alpha\} = 1-\alpha$ for $0<\alpha\le 1$

clearly measurable

$m\{x : f(x) > \alpha\} = 0$ for $\alpha > 1$

still measurable

$f(x)$ is measurable.

$f(x)$ is a Borel measurable function $\equiv$ For every α = real # or $\pm\infty$, $\{x: f(x) > \alpha\}$ is a <u>Borel</u> set.

ex: A Borel measurable function is a measurable function.

Q. Does one encounter nonmeasurable functions in practice?

A. <u>No</u>.

It is quite difficult to concoct a nonmeasurable set or nonmeasurable function. They are abstractly defined and are *pathological* cases. The standard example is defined point by point:

Consider real numbers in $[0,1]$. Divide $[0,1)$ into an infinite number of sets as follows:

If s and t are real numbers and $s-t$ is a rational number (ratio of integers), then s and t are in the same set.

For example, one set contains all the rational #'s between 0 and 1: $\{0, 1/2, 1/3, 1/4, 2/5, 2/3, 3/4, \ldots\}$

Another set contains $\pi/4 \pm$ rationals: $\{\pi/4, \pi/4 - 1/2, \pi/4 + 1/8, \ldots$

Another set contains $1/\sqrt{2} \pm$ rationals: $\{1/\sqrt{2}, 1/\sqrt{2} + 1/4, 1/\sqrt{2} - 1/2, \ldots$

Now create a set S which contains exactly one element from each of the above sets:

$S = \{0, \pi/4, 1/\sqrt{2}, \ldots\}$

Then S is not measurable.

pf: Create sets $S \oplus r_i$ by adding rational number r_i to every element of S. Use modulo one arithmetic: if $s + r_i > 1$ then result should be $s + r_i - 1$. (over)

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Now $\bigcup_{\substack{r_i \\ \text{rational}}} S \oplus r_i = [0,1)$. Union contains all pts in $[0,1)$.

Also $S \oplus r_i \cap S \oplus r_j = \emptyset$ for $i \neq j$. The sets do not intersect.

and all the $S \oplus r_i$ must have the same measure because they all have the same number of pts which are distributed the same way in $[0,1)$.

Then $m\underbrace{\left(\bigcup_{r_i} S \oplus r_i\right)}_{} = \sum_{r_i} m(S \oplus r_i)$ since $S \oplus r_i$ do not intersect.

$= m[0,1) = 1$

Since there infinitely many r_i and every $S \oplus r_i$ has the same measure we have:

$$1 = m[0,1) = \sum_{r_i} m(S + r_i) = \infty \cdot m(S)$$

If $m(S) = 0$ then we must have $\infty \cdot 0 = 1$, but usually we define $\infty \cdot 0$ to be 0 in real analysis. If $m(S) \neq 0$ we would get $\infty \cdot \text{const} = \infty$, but this is wrong since we must get $m[0,1) = 1$.

We conclude that S is <u>not measurable.</u>

Moral: Nonmeasurable Sets are uncommon and may be viewed as abstract curiosities. In the above example we found a product $\infty \cdot 0$ making an appearance. What is really at the heart of problems where $\infty \cdot 0$ appears is always the issue of cardinality (i.e. sizes) of sets. The set of real #'s is much larger than the set of rational #'s, and the rationals are already infinite in number. We had infinitely many $S + r_i$, each containing infinitely many points