EX: Fig. 1 below shows an example of how convolution may be done with the Rosetta Stone method wherein functions are differentiated until they become impulses, the impulses are convolved with other functions—resulting in the replication of those other functions—and the results are integrated to obtain the final result.

The key identities for these operations are the following:

1) $f(t)*g(t)=g(t)*f(t)$ — Commutativity

2) $f(t)*g(t)=\int_{-\infty}^{t}\frac{df(t)}{dt}*g(t)dt$ — Derivative of convolution

3) $A\delta(t-a)*f(t)=Af(t-a)$ — Replication by impulses

The Rosetta Stone convolution worksheet is convenient for finding convolutions graphically.

Fig. 1 shows the convolution worksheet used for this example.

Notes, line by line on the solution in Fig. 1:

① The convolved functions are shown with the answer left blank.

② Either of the functions in Line ① may be differentiated, but differentiating the function on the left, $f(t)$, is simpler. We split $f(t)$ into two functions, $f_1(t)$ (an impulse) and $f_2(t)$ (a rectangle) that we convolve separately and then sum. This an example of superposition. Function $f_1(t)$ is on Line ② and is ready for the convolution operation, yielding the result in the right column.

③ The rectangle is a step up and a step down, which is differentiated to get a negative impulse function of area −2, f'_{21}, and a positive impulse function of area +2, f'_{22}. The impulses are processed separately. Each impulse creates a copy of g(t), shifted by the position of the impulse and multiplied by the area of the impulse.

④ The second impulse, f'_{22}, replicates $g(t)$ right-side up and shifted to the right by one-half unit.

⑤ The results of convolution, on the right side of Lines ③ and ④, are summed. Then that sum is integrated (darker function). The slanted (or linear) portions of the summed results on Lines ③ and ④ become quadratic (or parabolic) segments in the integral. Note that the integral of a polynomial increases its order by one: a constant integrates to a sloped line, a sloped line integrates to a quadratic, etc. Fig. 2 shows the first quadratic segment, from -1 to -0.5, in the final function. To sketch such a quadratic segment, we use the following information:

1) The slope of an integral at any point is the value of the function being integrated at that point. We use the values of the function being integrated at the endpoints of the quadratic segment to set the slopes of the final result at the endpoints. The lighter curve in Line ⑤, shown in expanded view in Fig. 2, is being integrated, and it has values of $-1/8$ at $t = 1$ and 0 at $t = 1.5$. Thus, the integral (darker line) has slopes of $-1/8$ at $t = 1$ $t = 1.5$ and 0 at $t = 1.5$. The slopes are similar, so the segment of the integral looks almost linear.

2) The change in the value of an integral between two points in time is the same as the area under the function being integrated between the two points in time. Between $t = 1$ and $t = 1.5$ in Fig. 2, the lighter curve has an area of 1/8. Thus, the quadratic segment must have a vertical change of 1/8 between $t = 1$ and $t = 1.5$. In Fig. 2, the integral changes from $-1/8$ to 0.

3) The sketch of a quadratic segment between endpoints may be approximated in most cases with little loss of accuracy if the above rules are observed at the endpoints.

⑥ The sum of Line ⑤ and Line ② is the value of the convolution, $f(t) * g(t)$ in Line ⑥. Note that an impulse carries through to the sum unchanged. For a consistency check, the convolution of two functions will be of order one more than the sum of the orders of the functions. A constant segment has order zero. So the convolution of two constant segments has order one (linear slope). The convolution of a constant segment and a sloped segment has order $0 + 1 + 1 = 2$, which is quadratic. Curiously, an impulse may be thought of as having an order of -1, which is another impulse. The rule applies to all these cases.

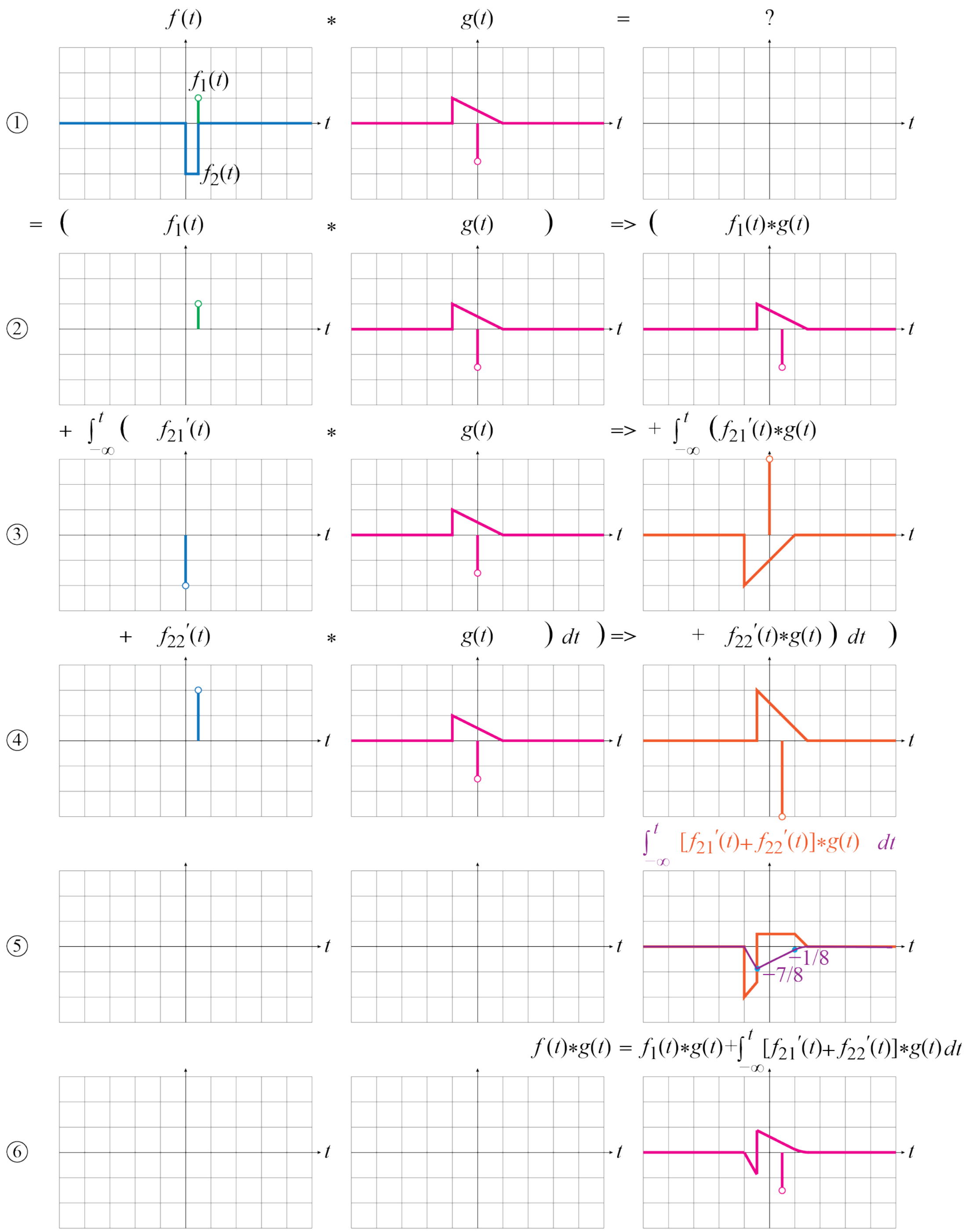

Fig. 1. Convolution by differentiation, convolution, integration.

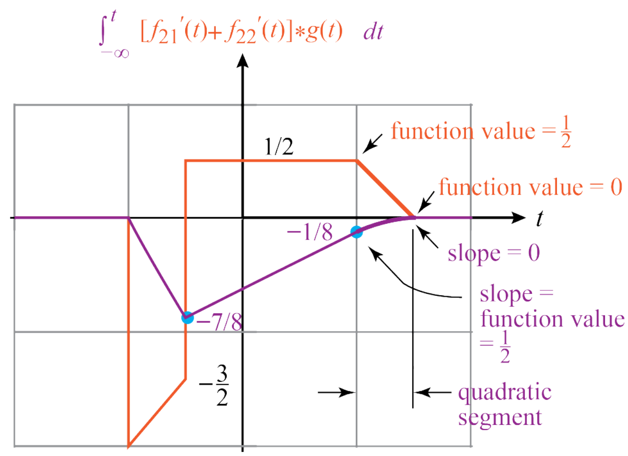

Fig. 2. Expanded view of quadratic segment in final convolution.

Darker line is the integral.