TOOL: The derivation below steps through equations for the linear transformation of an example function, $x(t)$, shown in Fig. 1.

$$x(t) = \begin{cases} 0 & -\infty < t \le -1 \\ \frac{2}{3}(t-1) & -1 < t < 2 \\ 0 & 2 < t < \infty \end{cases}$$

NOTE: The graphical method in CTools: Rosetta Stone Method: Graphical Math: Function Transformations: Procedure is much faster than the equation approach presented here and is recommended as the preferred method of transforming functions. For those who like equations, proceed.

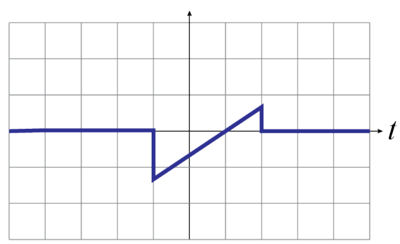

Fig. 1. Example function, $x(t)$, for transformation.

The generic form of a linear function transformation is $y(t) = ax(b[t-c])$. The transformation is said to be linear because the basic shape of the function stays the same.

The example transformation we examine here is, $y(t) = -2x(-1.5t-3)$. Note that the transformation of $x(t)$ involves replacing the argument of $x(t)$, which is t, with a function $g(t) = -1.5t-3$. Thus, we have $y(t) = -2x(g(t))$. We may pick a value of t, evaluate $g(t)$, and substitute that value of $g(t)$ for t in the definition of $x(t)$ and multiply by -2 to find the value of $y(t)$. That substitution is the basis of the derivation below. Things get complicated owing to the piecewise definition of $x(t)$.

We start with the definition of $x(t)$, repeated here.

$$x(t) = \begin{cases} 0 & -\infty < t \le -1 \\ \frac{2}{3}(t-1) & -1 < t < 2 \\ 0 & 2 < t < \infty \end{cases}.$$

If we wish to express $y(t) = ax(b[t-c])$ as a function, we replace t with $b[t-c]$ in both the function values and the intervals of the definition, and we multiply the function value by a.

$$y(t) = \begin{cases} 0 & -\infty < -1.5[t-(-2)] \le -1 \\ (-2)\frac{2}{3}(-1.5[t-(-2)]-1) & -1 < -1.5[t-(-2)] < 2 \\ 0 & 2 < -1.5[t-(-2)] < \infty \end{cases}$$

After this, we simplify the definition. The conditions on time intervals are tedious.

First, we multiply the inequalities by -1, flipping their direction, so t will appear as a positive quantity.

$$y(t) = \begin{cases} 0 & \infty > 1.5[t+2] \ge 1 \\ 2t + \frac{16}{3} & 1 > 1.5[t+2] > -2 \\ 0 & -2 > 1.5[t+2] > -\infty \end{cases}$$

Second, we to write the inequalities in conventional order (smaller value on the left).

$$y(t) = \begin{cases} 0 & 1 \le 1.5t+3 < \infty \\ 2t + \frac{16}{3} & -2 < 1.5t+3 < 1 \\ 0 & -\infty < 1.5t+3 < -2 \end{cases}$$

Third, we subtract 3 from each term in the inequalities in order to isolate t.

$$y(t) = \begin{cases} 0 & -2 \le 1.5t < \infty \\ 2t + \frac{16}{3} & -5 < 1.5t < -2 \\ 0 & -\infty < 1.5t < -5 \end{cases}$$

Fourth, we divide by 1.5 to leave t by itself in the middle of the inequalities. Note: if we had to divide by a negative value, we would flip the directions of the inequalities, as we did in the first step.

$$y(t) = \begin{cases} 0 & -\frac{4}{3} \le t < \infty \\ 2t + \frac{16}{3} & -\frac{10}{3} < t < -\frac{4}{3} \\ 0 & -\infty < t < -\frac{10}{3} \end{cases}$$

Fifth, we reorder the inequalities so the leftmost interval is listed first.

$$y(t) = \begin{cases} 0 & -\infty < t < -\dfrac{10}{3} \\ 2t + \dfrac{16}{3} & -\dfrac{10}{3} < t < -\dfrac{4}{3} \\ 0 & -\dfrac{4}{3} \leq t < \infty \end{cases}$$

Fig. 2 shows the graph of $y(t)$. One graph is worth 100 equations...

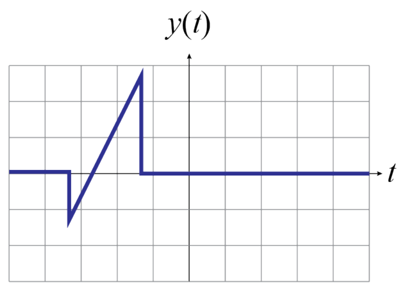

Fig. 2. $x(t)$ and $y(t) = -2x(-1.5t - 3)$.