TOOL: The following rules govern precise graphing of $\dfrac{df(t)}{dt}$ given a quadratic shape of $f(t)$ in a given interval, $[t_1, t_2]$. Fig. 1 shows an example function that illustrates the process.

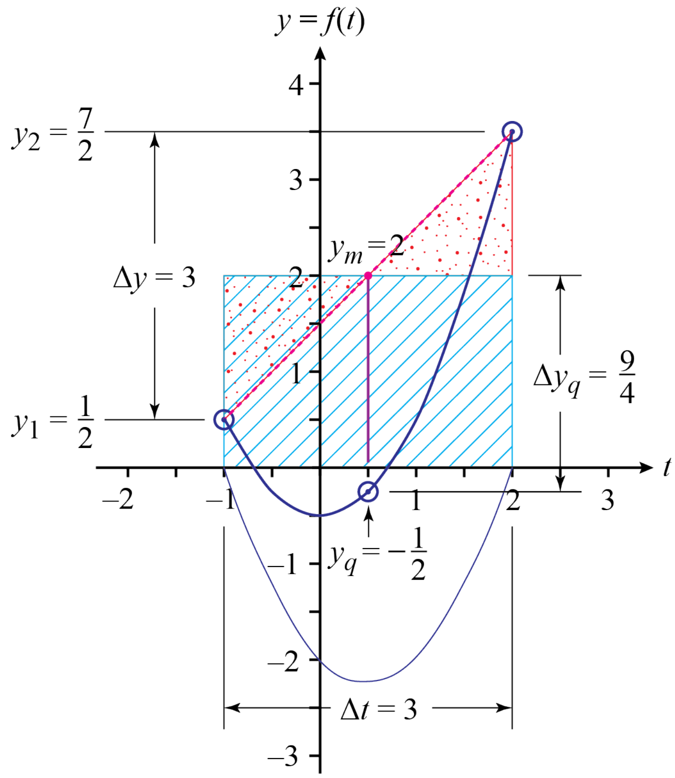

Fig. 1. Example function for process of graphing derivative of quadratic segment.

<u>Steps for sketching derivative of quadratic</u>

1. Find the values of the quadratic at two endpoints and a midpoint:

$$y_1 = f(t_1) \qquad y_q = f\left(\frac{t_1 + t_2}{2}\right) \qquad y_2 = f(t_2).$$

2. Draw a line from y_1 to y_2 and mark the midpoint as $y_m = \dfrac{y_1 + y_2}{2}$. Find the distance, Δy_q from y_q to y_m:

$$\Delta y_q = y_m - y_q$$

3. Calculate the following values:

$$\Delta t = t_2 - t_1 \qquad \Delta y = y_2 - y_1 \qquad m = \frac{\Delta y}{\Delta t} \qquad q = \frac{\Delta y_q}{\Delta t}$$

4. The derivative of the quadratic is a straight line that connects the following two points:

$$(t_1, y_1) = m - 4q \qquad (t_2, y_2) = m + 4q$$

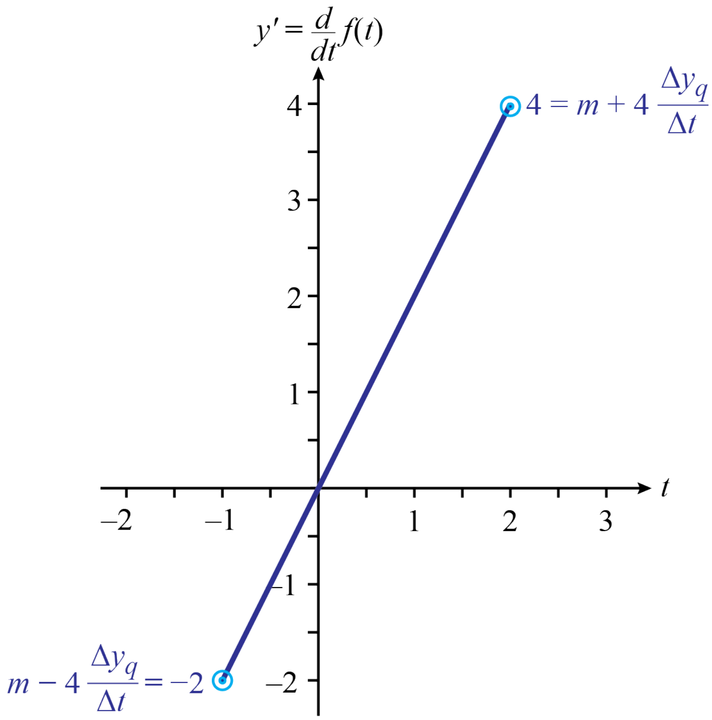

Fig. 2. Derivative of example function using above process.

DERIV: The following derivation is optional reading and is provided only for completeness.

A quadratic function segment may be decomposed into a constant function, a linear function, and a quadratic function, as shown in Fig. 3.

$$y(t) = y_m + m(t - t_m) + \Delta y_q \left(\frac{t - t_m}{\Delta t/2} \right)^2 - \Delta y_q \tag{1}$$

where $t_m = \dfrac{t_1 + t_2}{2}$.

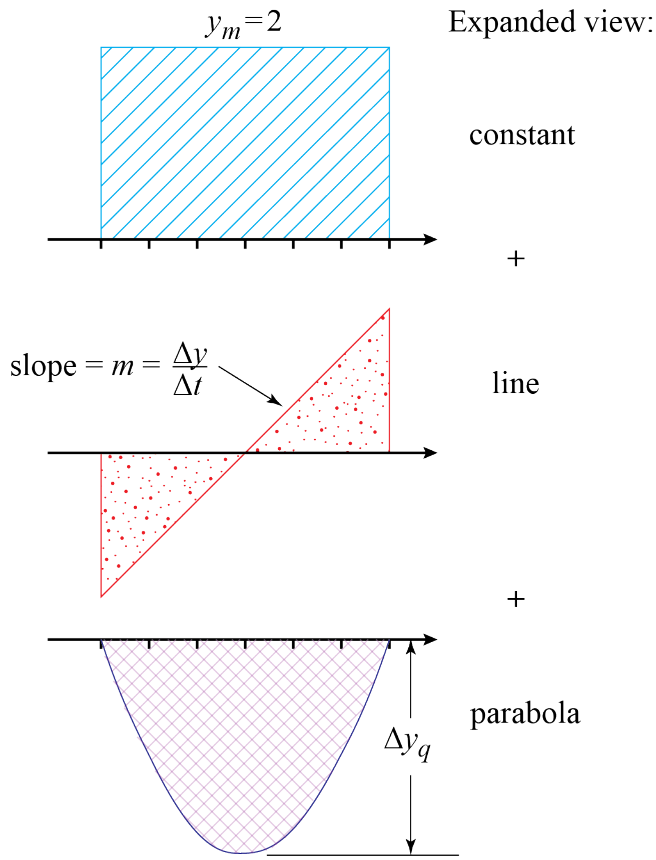

Fig. 3. Decomposition of quadratic segment into a sum of components of increasing order.

Assuming the expression in (1) is correct, which we verify later on, we have the following derivative formula:

$$\frac{dy(t)}{dt} = m + 2\Delta y_q \left(\frac{t - t_m}{(\Delta t/2)^2} \right). \tag{2}$$

The derivative is a straight line segment. Evaluating (2) at t_1 and t_2, we have the endpoints of the line segment.

$$\left.\frac{dy(t)}{dt}\right|_{t=t_1} = m - 4\frac{\Delta y_q}{\Delta t}. \tag{3}$$

$$\left.\frac{dy(t)}{dt}\right|_{t=t_2} = m + 4\frac{\Delta y_q}{\Delta t}. \tag{4}$$

where we use

$$t_1 - t_m = -\Delta t/2 \text{ and } t_2 - t_m = \Delta t/2. \tag{5}$$

To verify that (1) is equivalent to the formula for the quadratic segment, we show that it produces correct values at three points: (t_1, y_1), (t_m, y_m), and (t_2, y_2). If a quadratic function matches another quadratic function at three points, the two quadratic functions are the same. In other words, a quadratic function may specified by three distinct points on the quadratic function.

$y(t = t_1)$:

$$y(t_1) = y_m + m(t_1 - t_m) + \Delta y_q \left(\frac{t_1 - t_m}{\Delta t/2} \right)^2 - \Delta y_q$$

or, using (5),

$$y(t_1) = y_m + m(-\Delta t/2) + \cancel{\Delta y_q \left(\frac{-\Delta t/2}{\Delta t/2} \right)^2} - \cancel{\Delta y_q}$$

or, substituting for y_m, m, and Δt,

$$y(t_1) = \frac{y_1 + y_2}{2} + \frac{y_2 - y_1}{\Delta t}(-\Delta t/2) = y_1 \checkmark$$

By symmetry, we have

$$y(t_1) = \frac{y_1 + y_2}{2} + \frac{y_2 - y_1}{\Delta t}(\Delta t/2) = y_2 \checkmark$$

$y(t = t_m)$:

$$y(t_m) = y_m + m\cancel{(t_m - t_m)} + \cancel{\Delta y_q \left(\frac{t_m - t_m}{\Delta t/2} \right)^2} - \Delta y_q$$

or

$$y(t_m) = y_m - \Delta y_q = y_q \checkmark$$