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TOOL: The Rosetta Stone method reveals the relationship of the Fast Fourier Transform (FFT) to the Fourier transform of a continuous-time signal before it is sampled.

The FFT is just an efficient calculation of a Discrete Fourier Transform (DFT). The DFT, which might be more appropriately called a Discrete-Frequency discrete-time Fourier Transform (DFFT) of $f(t)$ is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

where n is an index for a window of N samples of sampled waveform $x[n]$. The values of k run from 0 to $N-1$.

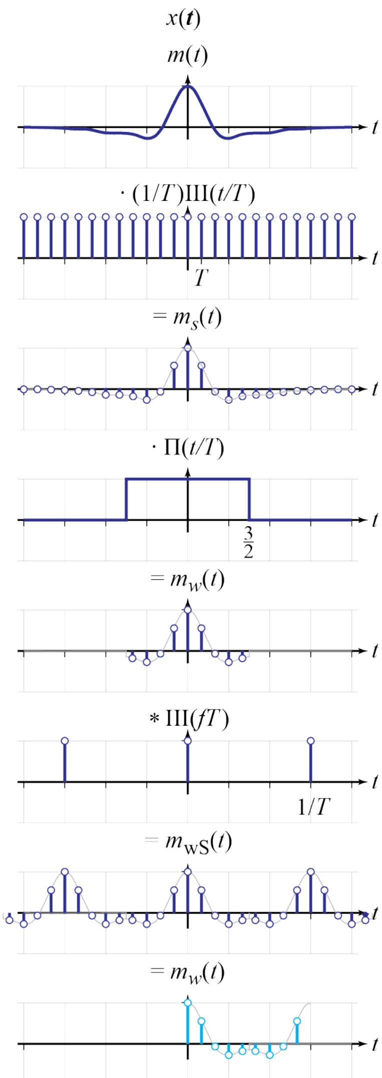
The efficiency of the FFT results from eliminating redundancies in values of the root of unity, $e^{-j2\pi kn/N}$, in the DFT formula.

The FFT may be seen as a Discrete-Time Fourier Transform (DTFT) sampled in the frequency at N frequencies between 0 and the sampling frequency, f_s. Recall that the DTFT is the Fourier transform of a sampled waveform and, although continuous in the frequency domain, repeats at multiples of the sampling rate. Thus, all information from the DTFT is found in the interval from 0 to f_s in the frequency domain.

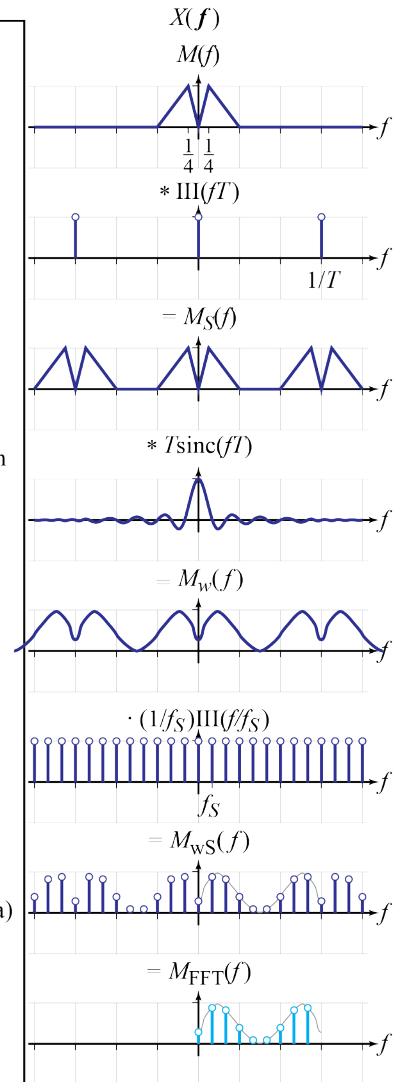
Sampling in the frequency domain, however, based on a window of only N samples from $x[n]$ is bound to give an incomplete picture of the spectrum $X(f)$. The example Rosetta Stone analysis shows what happens to the spectrum in an FFT.

In Lines 1 to 3, we create the sampled version of a band-limited signal, $m(t)$. In Line 4, a window of data is extracted, causing a convolution in the frequency domain that results in distortion of the spectrum.

In Line 6, we sample one period of the spectrum, causing the windowed values in the time domain to be repeated. Remembering that we capture a waveform's complete spectrum only if we sample at twice its highest frequency, the shape of the spectrum (now thought of as a time-domain waveform) will determine how well the sampled spectrum captures the true shape of the continuous spectrum we have sampled in the frequency domain. We retain all of the spectrum if and only if we have no aliasing in time. In other words, we retain all of the spectrum if and only if our window of data captured all the nonzero values of $m(t)$.



Time-domain waveform = $(4/3)\text{sinc}^2(t)$ $-(1/3)\text{sinc}^2(t/4)$	Band-limited signal = $(4/3)\Lambda(t)$ $-(4/3)\Lambda(4f)$
Sampling rate 3 per time unit avoids aliasing	Spectrum copies will be at $f = 3n$
sampled waveform	Sampled spectrum
Multiply by rect function to extract window of data	Convolve with spectrum of rect function
only samples inside window remain	Spectrum is distorted, spreads out
convolve windowed data with impulses	Sample the spectrum
windowed data replicated	Sampled spectrum (of replicated data)
IFFT recovers <u>undistorted</u> windowed data	Copy of spectrum is <u>distorted</u> and is FFT
windowed data replicated ≠ original waveform	FFT gives spectrum of replicated data



Line 7 shows what the FFT captures in cyan color. Line 8 just serves to emphasize that the FFT and its inverse are a one-to-one mapping. When we take the inverse FFT, we get our original samples back.

<u>Conclusion</u> We get distortion from either windowing $m(t)$ or from aliasing when we sample m(t), or both. Higher sampling rates and longer windows help, as expected.

REF: John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications,* 3rd Ed, Upper Saddle River, NJ: Prentice Hall, 1996.