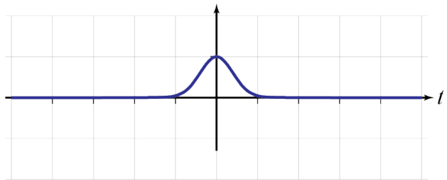
**Ex:** Complete the Rosetta Stone diagram on the next page, below. This example shows sampling of a continuous-time waveform achieved by multiplying by an impulse train. The sampled waveform is then AM modulated, which is not a typical operation for a sampled waveform, but the Rosetta Stone method tells us what the result of this AM modulation will be.

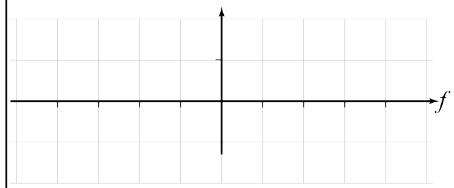
After the AM modulation, the signal is filtered in Line ⑥. The impulse response of the filter is a sinc function. Although this impulse response is non-causal, it may be approximated by truncating and delaying a causal impulse response.

See diagram next page.

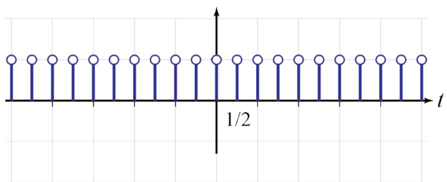
$e^{-\pi t^2}$



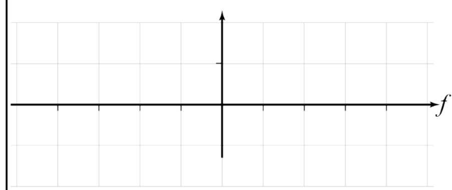

① $t$-domain waveform = gaussian



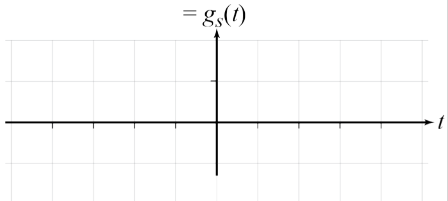

$\cdot 2\text{III}(t/2)$



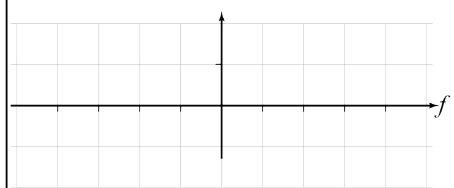

② Impulse train for sampling



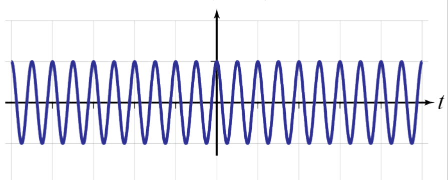

$= g_s(t)$



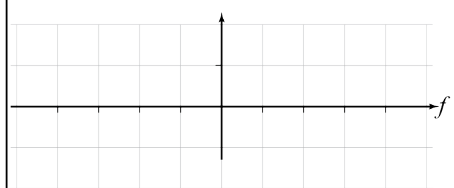

③ Sampled waveform



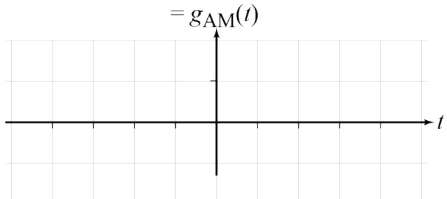

$\cdot\ \cos(2\cdot 2\pi t)$



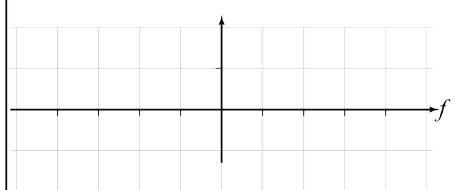

④ AM modulation



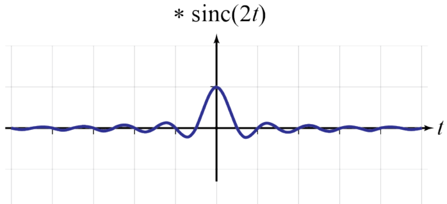

$= g_{\text{AM}}(t)$



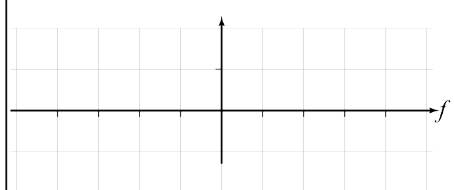

⑤ =gs(t) since cos = 1 at samples



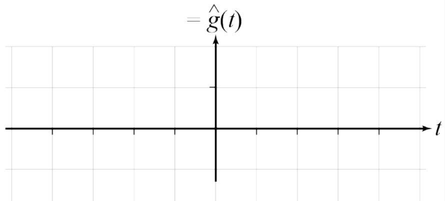

$*\ \text{sinc}(2t)$




⑥ impulse response of low-pass

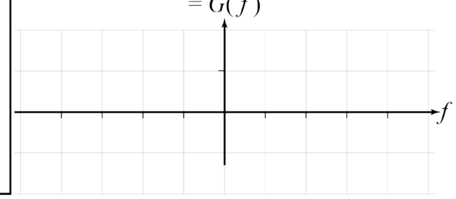



$= \hat{g}(t)$




⑦

$= \hat{G}(f)$



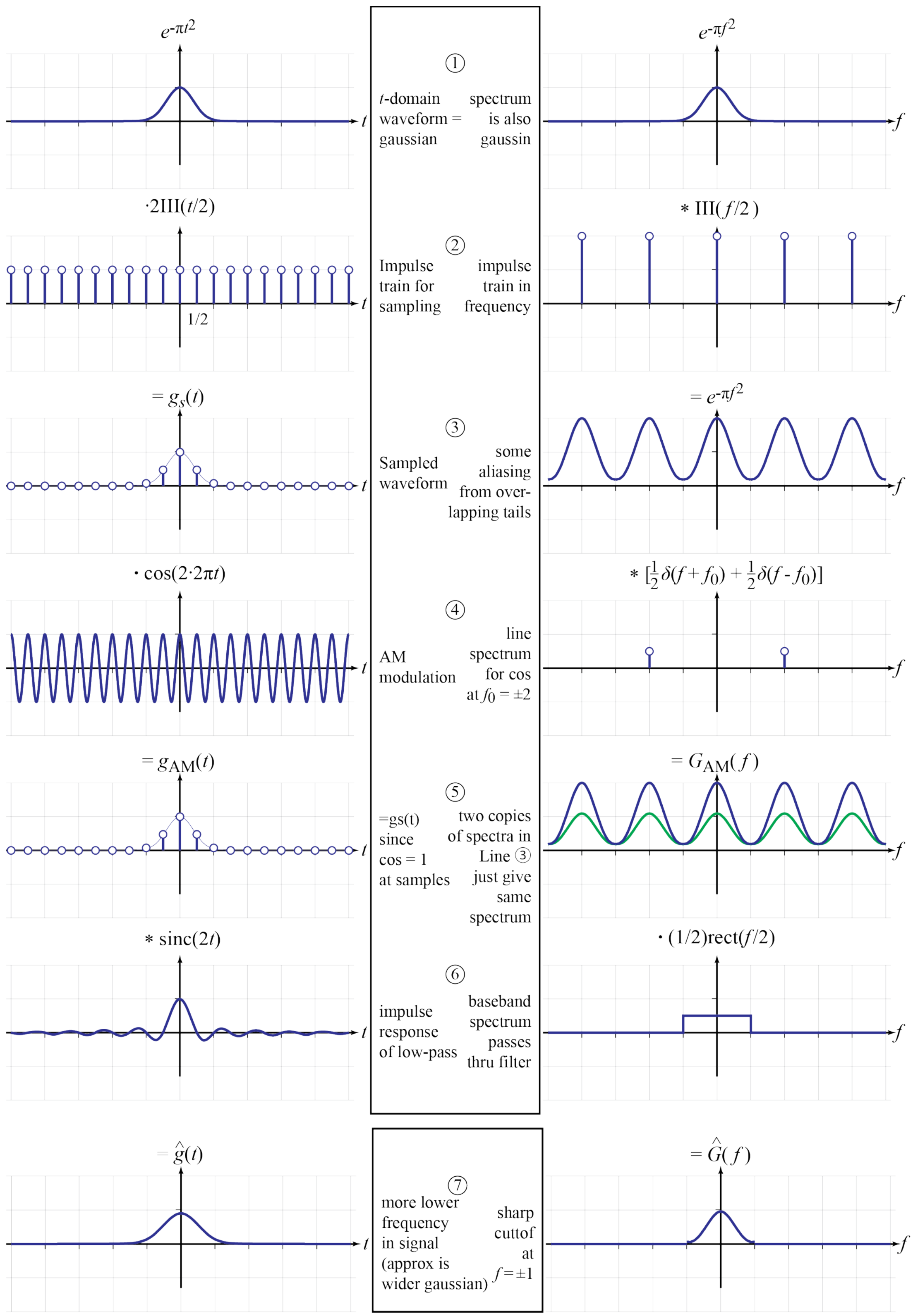
**SOLN:** A sampled waveform is equivalent to multiplying the waveform by an impulse train. The result of sampling in this example is shown on Line ③. In the frequency domain, the spectrum of the original signal is replicated by convolving with an impulse train (spaced by the sampling frequency). This repeated spectrum is also what a Discrete-Time Fourier Transform (DTFT) gives us (using sampling values rather than impulses). Thus, the Rosetta Stone approach, based on only the standard Fourier transform, accounts for what the DTFT tells us.

In this example, the multiplies on the left side (time) result in convolutions on the right side (frequency). Convolution of an impulse and a waveform results in a copy of the waveform with the origin shifted to the location of the impulse. Thus, convolution of an impulse train with a waveform (a spectrum here) creates infinitely many copies of that waveform (spectra here). Note that, if we wish to inverse transform a repeated spectrum, we would use and inverse DTFT:

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(e^{j2\pi f})e^{j2\pi fn}df.$$

In Line ⑥, where the sampled signal is turned back into a continuous-time waveform, the impulse response is a sinc function in time. This corresponds to an ideal low-pass filter, which is a rectangle function in the frequency domain. Assuming an ideal filter is convenient. In practice, the result of using a realistic filter gives similar result but creates a messy and complex diagram. The approximation of using an ideal filter allows us to get a fairly accurate result that is remarkably simple.

The spectrum of the final output of the system in this problem is a gaussian shape that cuts off sharply at $f = \pm 1$. It is difficult to say what the time-domain waveform at final line should be, but a rough approximation is to widen the waveform and reduce the amplitude. This approximation assumes that a signal with less high-frequency content will look smoothed out. If a better approximation is needed, a numerical approximation of the inverse Fourier transform would be used. That is, we would use small rectangles to approximate the inverse Fourier transform, just like the example of integration as a sum of rectangles that is given in calculus classes.


$e^{-\pi t^2}$
① t-domain waveform = gaussian
spectrum is also gaussin
$e^{-\pi f^2}$
$\cdot 2\text{III}(t/2)$
1/2
② Impulse train for sampling
impulse train in frequency
$* \text{ III}(f/2)$
$= g_s(t)$
③ Sampled waveform
some aliasing from over-lapping tails
$= e^{-\pi f^2}$
$\cdot \cos(2\cdot 2\pi t)$
④ AM modulation
line spectrum for cos at $f_0 = \pm 2$
$* [\frac{1}{2}\delta(f+f_0) + \frac{1}{2}\delta(f-f_0)]$
$= g_{AM}(t)$
⑤ =gs(t) since cos = 1 at samples
two copies of spectra in Line ③ just give same spectrum
$= G_{AM}(f)$
$* \text{ sinc}(2t)$
⑥ impulse response of low-pass
baseband spectrum passes thru filter
$\cdot (1/2)\text{rect}(f/2)$
$= \hat{g}(t)$
⑦ more lower frequency in signal (approx is wider gaussian)
sharp cutof at $f = \pm 1$
$= \hat{G}(f)$
t
f

**REF:** Ronald Bracewell, *The Fourier Transform and its Applications*, 2nd Ed., New York, NY: McGraw-Hill, 1978. ISBN 0-07-007013-X

**SOLN:** To find $M_{Het}(t)$ in Line 5, we just replicate the spectra with their origins translated to $\pm f_0 = \pm 1.5$. In proper AM demodulation, spectral copies at the origin reinforce. Here they end up side-by-side.

For the spectrum in the last line of the diagram, we multiply the functions in Line 5 and Line 6. This operation low-pass filters the signal, and we get a triangle wave, which is nothing like the original spectrum.

The inverse transform of the triangle waveform is a $\text{sinc}^2$ function. This is a standard Fourier transform pair that is symmetrical since the functions are even and real.

At first glance, the final waveform looks similar to the original $m(t)$, but the central peak is higher, and the zero-crossings are different. Part of the spectrum has been eliminated by the erroneous AM demodulation.

**REF:** Ronald Bracewell, *The Fourier Transform and its Applications*, 2nd Ed., New York, NY: McGraw-Hill, 1978. ISBN 0-07-007013-X