CONCEPT: The computation of transfer functions from poles and zeros differs for analog and digital Linear, Time-Invariant (LTI) filters. For analog filters, we use poles and zeros and values of $s = j\omega$ on the imaginary axis to compute $H(j\omega)$. For digital filters, we use poles and zeros and values of $z = e^{j\omega}$ on the unit circle to compute $H(e^{j\omega})$. In both cases the transfer function is a complex-valued function. The product of the transfer function and the phasor value of an input sinusoid at one frequency gives the phasor value of the output sinusoid for that frequency. By embedding sampled waveforms in a continuous-time framework, the Rosetta Stone method offers an alternate method of finding the transfer function that lacks the unit circle view. The discussion below explores the relationship between the unit-circle approach and the Rosetta Stone method.

NOT'N: Before starting our analysis, it is helpful to define the various transforms under discussion. In the text that follows, the transforms of a given signal or impulse response are distinguishable by the argument of the transformed function.

Name	Argument	Transform
Laplace	s	$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$
Fourier	f	$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \quad = X(s = e^{-j2\pi f})$
Z	z	$X(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$
DTFT	$e^{j\omega}$	$X(e^{j2\pi f}) \equiv \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}dt \quad = X(z = e^{j2\pi f})$

Note that transforms for frequency response are 2-sided, but 2-sided and 1-sided transforms give the same result for causal impulse responses (with zero initial conditions).

We start with the standard method for finding the transfer function, $H(e^{j\omega})$, of a digital filter. That is, we express $H(e^{j\omega})$ as a ratio of zero terms over pole terms. Fig. 1 shows a simple digital filter with one pole. The impulse response of the filter is an exponential decay.

$$h[n] = e^{-n}u[n] \tag{1}$$

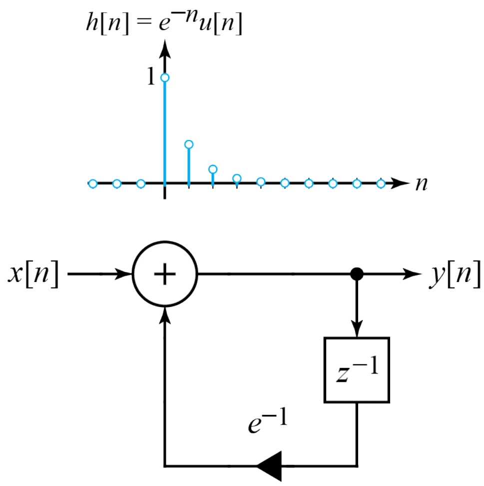

Fig. 1. One-pole digital filter with decaying exponential impulse response.

The impulse response may be verified by setting initial conditions to $y[-1]=0$ and $x[n]=\delta[n]=[\ldots,0,\underline{1},0,\ldots]$, where underscore indicates $n=0$ sample. Then we run the filter, which results in the previous output value being cycled around and divided by e each time.

$$h[n]=e^{-n}u[n] \tag{1}$$

To find the transfer function using the Z-transform, we write the difference equation for the filter.

$$y[n]=x[n]+e^{-1}y[n-1] \tag{2}$$

For the sake of simplicity, we take the sampling rate to be $f_s = 1\,\text{Hz}$ so normalized frequency and sampling rate are the same.

To find the frequency response of the filter we use the 2-sided Z-transform.

$$H(z)=\sum_{n=-\infty}^{\infty} h[n]z^{-n} \tag{3}$$

Applying the Z-transform to (2), we find the transfer function, $H(z)$, of the filter.

$$Y[z]=X(z)+e^{-1}z^{-1}X(z)$$

and

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1-e^{-1}z^{-1}}=\frac{z}{z-e^{-1}} \tag{4}$$

The filter has a zero at $z=0$ and a pole at $z=e^{-1}$, as shown in Fig. 2.

For the frequency response, we use $z = e^{j\omega} = e^{j2\pi f}$ on the unit circle. As f increases, z winds around the unit circle, and the values of $H(z) = H(e^{j2\pi f})$ are periodic, as shown in Fig. 3.

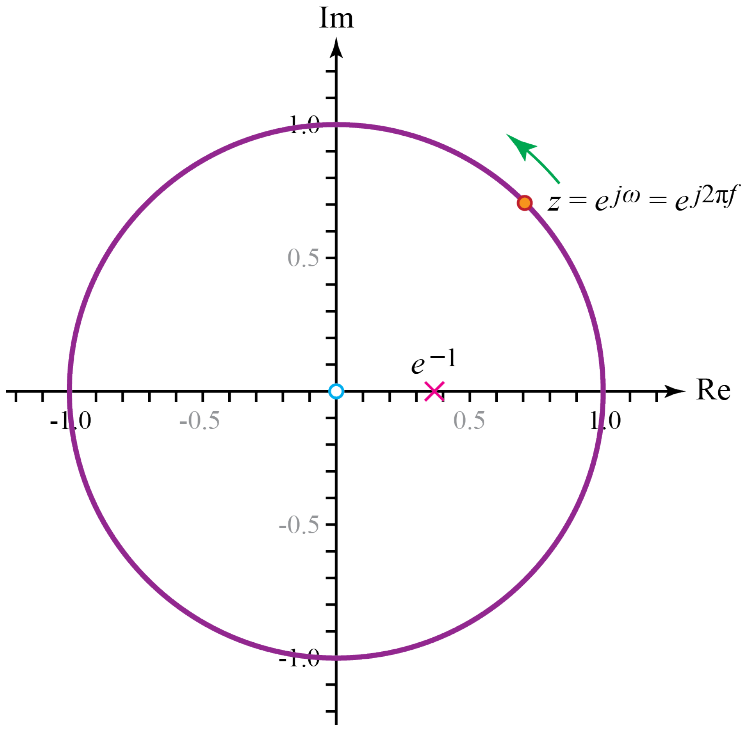

Fig. 2. Pole and zero of digital filter. Frequency response is found using $z = e^{j2\pi f}$.

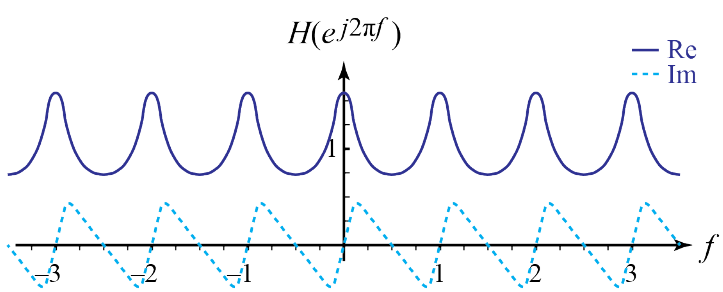

Fig. 3. Frequency response of digital filter is periodic.

As noted earlier, the frequency response, which is the Z-transform evaluated at $z = e^{j2\pi f}$, is the same as the DTFT of the sampled waveform.

$$H(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j2\pi fn} \qquad (5)$$

Using a geometric sum for (5) or using $z = e^{j2\pi f}$ in (4), we have the frequency response.

$$H(e^{j2\pi f}) = \frac{1}{1+e^{-1}e^{-j2\pi f}} \tag{6}$$

We now turn to an analysis of an analog filter to show how the Rosetta Stone method allows us to embed sampled systems in a continuous-time framework.

Consider the analog filter shown in Fig. 4 whose impulse response has the same shape as the digital filter presented above.

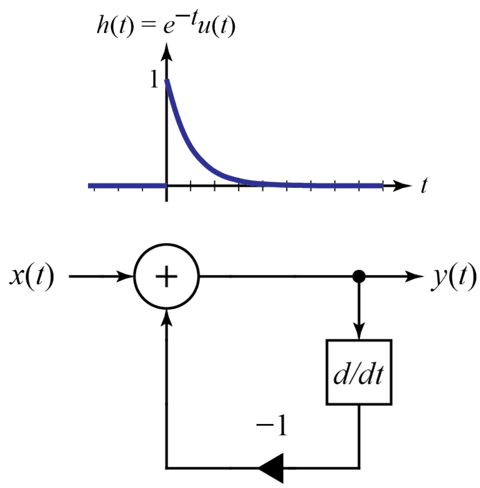

Fig. 4. One-pole analog filter with decaying exponential impulse response.

We use the Laplace transform to derive the impulse response for the analog filter. The time domain equation for the filter is

$$y(t) = x(t) - \frac{dy(t)}{dt}. \tag{7}$$

Laplace transforming (7) and doing some algebra yields the frequency response, $H(s = j2\pi f)$.

$$Y(s) = X(s) - sX(s)$$

and

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1+s} \tag{8}$$

The (causal) inverse Laplace transform is an exponential decay.

$$h(t) = e^{-t}u(t) \tag{9}$$

The filter has a pole at $s = -1$, as shown in Fig. 5.

For the frequency response, we use $s = j2\pi f$ moving up the imaginary axis. As f increases, the frequency response decays away, as shown in Fig. 6.

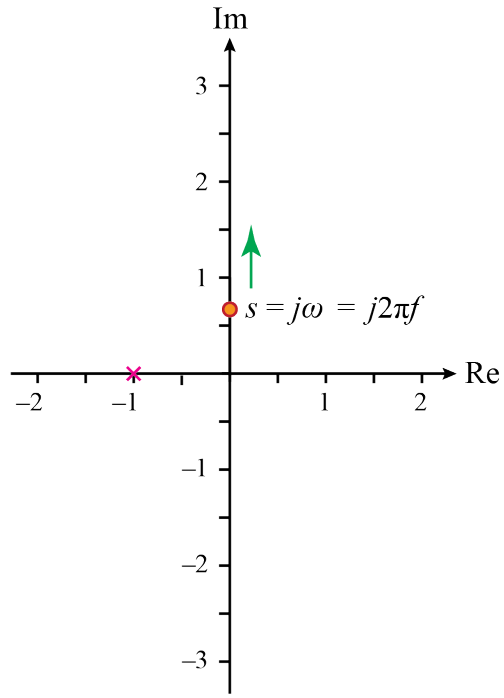

Fig. 5. Pole and zero of digital filter. Frequency response is found using $s = j2\pi f$.

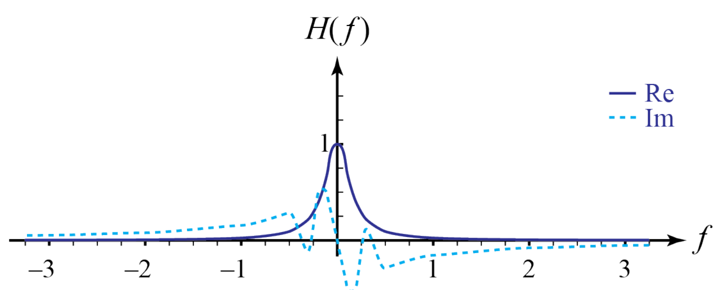

Fig. 6. Frequency response of analog filter is a low-pass.

If we sample the analog impulse response, $h(t)$, we expect to get the same $h[n]$ we have for the digital filter. We use Rosetta Stone method to explore this idea, as shown in Fig. 7.

The Rosetta Stone method employs the idea that a sampled waveform is equivalent to multiplying the continuous-time waveform by an impulse train in the time domain. In the frequency domain, the Fourier transform of the impulse train is an impulse train with impulses spaced by the sampling frequency. Multiplication in the time domain becomes convolution in the frequency domain, and convolving with an impulse creates a copy of a waveform. Thus, the frequency response is convolved by equivalent to replicating the frequency response at multiples of the sampling rate and summing.

On the diagram, Row 1 shows the exponential impulse response and its corresponding Fourier transform (i.e., frequency response). Row 2 shows the sampling impulse train and its Fourier transform, which is also an impulse train. Row 3 shows sampled impulse response (with the first sample highlighted in red) and the replicated frequency response waveform (thin lines) and sum (thick line, for clarity only real part is shown).

On Row 3, though, we have a problem: the sample of $h(t)$ at $t = 0$ has area 1/2 rather than the hoped for value of 1. Fig. 8 shows why: if we treat the impulse at $t = 0$ as the limit of rectangle functions that get get taller and narrower with a fixed area of one, the value of the product of the rectangles and the filter output approaches 1/2. Taking a closer look, with each rectangle as input, the filter output (the approximate impulse response) is seen to be an an exponential that rises to approximately one at $t = 0^+$. (The waveforms are familiar from from low-pass filters, for example an RC lowpass circuit). At $t = 0$, the filter output is seen to be approximately one-half. In the limit, the rectangles become an impulse, and the value of the sample value at $t = 0$ becomes exactly 1/2. In general, the sample value at a step discontinuity will be halfway up the step.

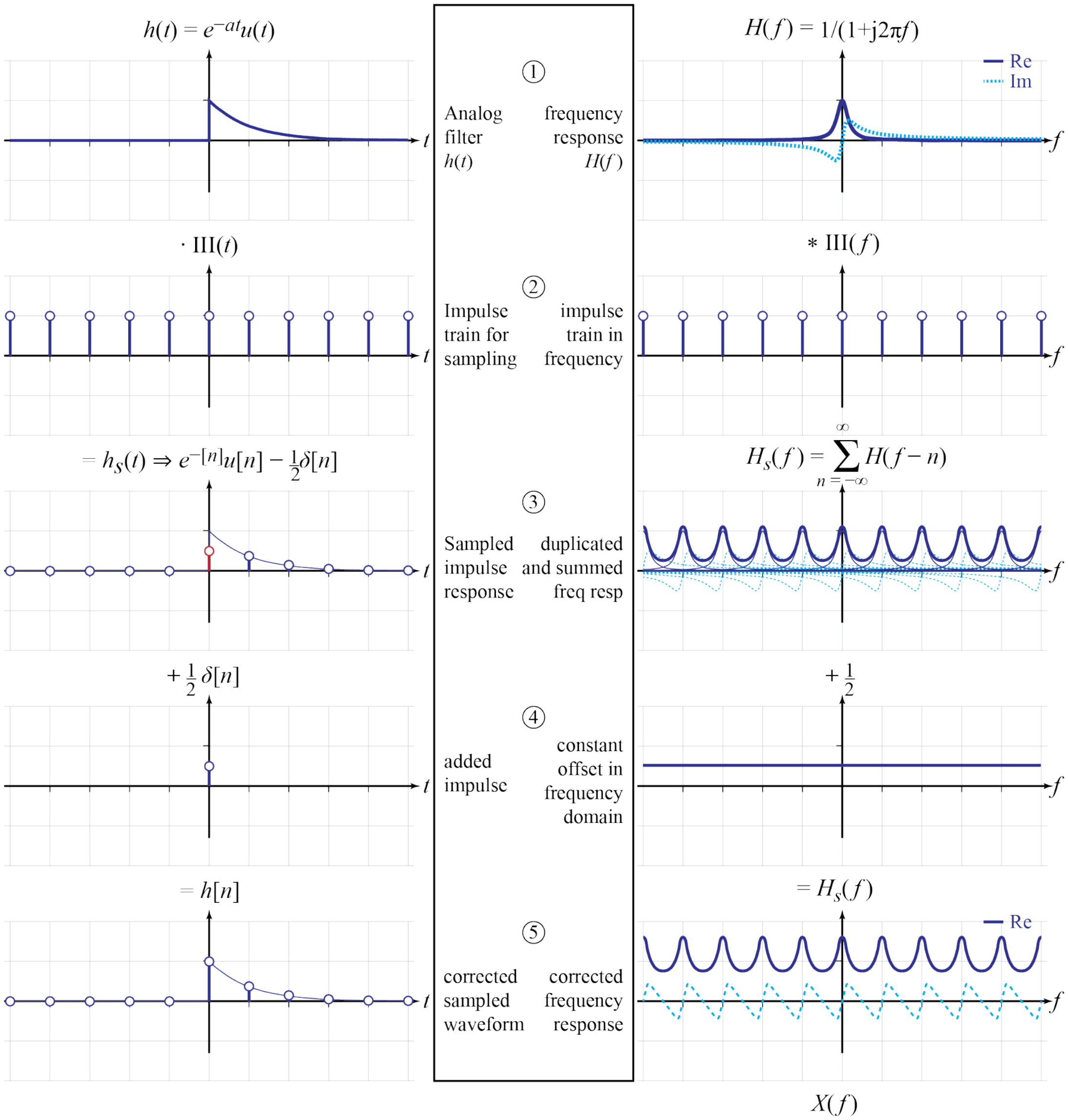

Fig. 7. Rosetta Stone diagram for sampling analog filter's impulse response.

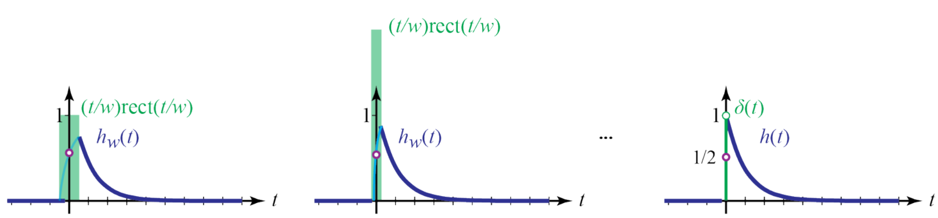

Fig. 8. Generalized function showing impulse at $t = 0$ as limit of rectangle functions.

Product of rectangles and $h(t)$ is 1/2 in limit as rectangle becomes an impulse.

Before we correct for the shortened impulse at $t = 0$, (which might be easily overlooked!), we get the replicated frequency response of the analog filter in the frequency domain on Row 3 of Fig. 7. This repetition may be interpreted as a filter bank each with each filter having a copy of the original pole duplicated every $j2\pi fn$ in the s-plane, as shown in Fig. 9. This would generalize to duplication of the original constellation of poles and zeros every $j2\pi fn$ for a more complicated filter.

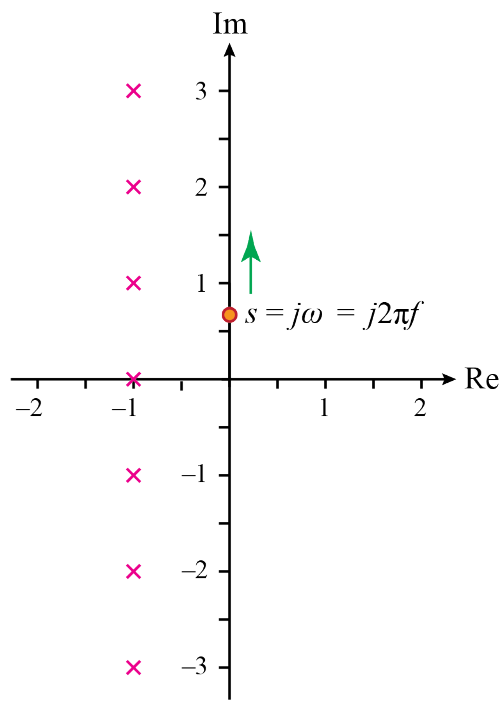

Fig. 9. Poles for filter bank are repeated every $j2\pi fn$ on the s-plane.

To correct for the shortened impulse at $t = 0$, we must add an impulse of height 1/2, which we do at Row 4 of Fig. 7. Note that this causes the frequency response curve to rise by 1/2 at all frequencies since the Fourier transform of an impulse is a constant. Also, note that there is no obvious way to modify h(t) to eliminate the need for the extra impulse. *The sampling process landing on a step-discontinuity created the need for the extra impulse.* Once we add the extra impulse, we get the correct frequency response on Row 5 of Fig. 7.

$$H_s(f) = H(e^{j2\pi f}) \tag{10}$$

In the final row of the Rosetta Stone diagram, because the frequency response is found using the Fourier transform, the final frequency response is shown as $H_s(f)$. This $H_s(f)$ is the sum of infinitely-many copies of the analog filter's frequency response. This stands in contrast to the approach taken earlier using the Z-transform, and we now demonstrate that $H_s(f) = H(e^{j2\pi f})$ using logic and by calculating the sum of the copies of $H(f)$.

Showing that the Fourier transform of the impulse train of the sampled $h(t)$ in the Rosetta Stone method is the DTFT of $h[n]$ is straightforward. The expression for the sampled $h(t)$ is a sum of impulses:

$$h[n] \text{ “=” } \sum_{n=-\infty}^{\infty} h(t=n)\delta(t-n) = \sum_{n=-\infty}^{\infty} h[n]\delta(t-n). \tag{11}$$

Using (11), we get the DTFT from the Fourier transform:

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} h[n]\delta(t-n) \right] e^{-j2\pi ft} dt \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} h[n]\delta(t-n) e^{-j2\pi ft} dt \\ &= \sum_{n=-\infty}^{\infty} h[n] e^{-j2\pi fn} \\ &= H(e^{j2\pi f}) \qquad \text{(DTFT)} \end{aligned} \tag{12}$$

Since we also get the DTFT from the Z-transform of $h[n]$, it must be the case that $H_s(f) = H(e^{j2\pi f})$. However, $H_s(f)$ is an infinite sum with an infinite axis for f, whereas $H(e^{j2\pi f})$ is computed from a single pole and the unit circle $e^{j2\pi f}$ for f. We now calculate the infinite sum in the Rosetta Stone method to verify equality of the two approaches.

The infinite sum in the Rosetta Stone method is

$$H(j2\pi f) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{1}{1 + j2\pi(f-n)} \tag{13}$$

A few steps of algebra transforms this sum into a standard form of summation found in references:

$$\begin{aligned} H(j2\pi f) &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{1+j2\pi(f-n)} \\ &= \frac{1}{2}+\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\dfrac{1+j2\pi f}{2\pi}-jn} \\ &= \frac{1}{2}+\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{a-jn} \end{aligned} \tag{14}$$

where $a \equiv \dfrac{1+j2\pi f}{2\pi}$.

We transform into a single-sided summation to ensure convergence.

$$\begin{aligned} \frac{1}{a-jn}+\frac{1}{a+jn} &= \frac{a+jn+a-jn}{a^2+n^2} \\ &= \frac{2a}{a^2+n^2} \end{aligned} \tag{15}$$

so

$$H(j2\pi f) = \frac{1}{2}-\frac{1}{2\pi a}+\frac{2a}{2\pi}\sum_{n=0}^{\infty}\frac{1}{a^2+n^2}. \tag{16}$$

From [1], we have the following summation formua:

$$\frac{1}{a^2+n^2} = \frac{\pi a\coth(\pi a)+1}{2a^2} \tag{17}$$

Substituting (17) into (16) and simplifying leads to an expression equivalent to $H(e^{j2\pi f})$:

$$\begin{aligned} H(j2\pi f) &= \frac{1}{2}-\frac{1}{2\pi a}+\frac{2a}{2\pi}\cdot\frac{\pi a\coth(\pi a)+1}{2a^2} \\ &= \frac{1}{2}+\frac{1}{2}\coth\pi a \end{aligned} \tag{18}$$

or

$$\begin{aligned} H(j2\pi f) &= \frac{1}{2}\left(1+\frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}\right) \\ &= \frac{1}{2}\left(\frac{1-e^{-2\pi a}}{1-e^{-2\pi a}}+\frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}\right) \\ &= \frac{1}{1-e^{-2\pi a}} \end{aligned} \tag{19}$$

or

$$\begin{aligned} H(j2\pi f) &= \frac{1}{1-e^{-2\pi a}} \\ &= \frac{1}{1-e^{-1}e^{-j2\pi f}} \\ &= H(e^{j2\pi f}) \end{aligned} \tag{19}$$

A remarkable fact that might be overlooked in the calculations is that Z-transforms involve geometric sums in terms of z^n, whereas the summation in the above derivation is in terms of n^2. Thus, the calculation of transfer functions using the unit circle is equivalent to very complicated infinite sums that are *not* geometric sums.

REF: [1] *WolframAlpha summation calculator*, https://www.wolframalpha.com/input?i=sum+calculator&assumption=%7B%22F%22%2C+%22Sum%22%2C+%22sumlowerlimit%22%7D+-%3E%220%22&assumption=%7B%22F%22%2C+%22Sum%22%2C+%22sumfunction%22%7D+-%3E%221%2F%281%2Ba*n%5E2%29%22&assumption=%7B%22F%22%2C+%22Sum%22%2C+%22sumvariable%22%7D+-%3E%22n%22&assumption=%7B%22F%22%2C+%22Sum%22%2C+%22sumupperlimit2%22%7D+-%3E%22infinity%22, accessed 16 July, 2026.