TOOL: One-way Analysis of Variance (ANOVA) tests the hypothesis that means are all the same for various treatments applied to a system, [1, 2].

ASSUMPTIONS:

1) There are k different treatments (samples) under consideration
2) For the ith of k different treatments (samples), there are n_i observations
3) All of the k treatments (samples) have the same variance, σ
4) The null and alternate hypotheses relate to equal means:

   H_0: $\mu_1 = \mu_2 = \ldots = \mu_k$

   H_1: At least one of the means is not equal to the others

DEFINITIONS:

α ≡ significance level for rejecting null hypothesis

k ≡ total number of samples (treatments) being considered

i ≡ index designating which sample (treatment) is being considered

N ≡ total number of observations (data points) available for all treatments

n_i ≡ number of observations available for sample (treatment) i

n ≡ number of observations available for each sample if all n_i are equal

j ≡ index designating which of n_i observations of sample i is being considered

μ ≡ grand mean of all observations for all samples

μ_i ≡ actual mean value for ith treatment

α_i ≡ difference between actual mean and grand mean for sample i: $\alpha_i = \mu_i - \mu$

ε_{ij} ≡ difference between jth observation of ith sample and μ_i: $\varepsilon_{ij} = x_{ij} - \mu_i$

x_{ij} ≡ value of observation i for sample (treatment) j: $x_{ij} = \mu + \alpha_i + \varepsilon_{ij}$

$\bar{x}_{i\bullet}$ ≡ calculated mean for all observations from ith sample (treatment)

$\bar{x}_{\bullet\bullet}$ ≡ calculated mean for all observations from all samples

SSA ≡ Sum of Squared errors of All treatment (sample) means vs grand mean

SSE ≡ Sum of Squared Errors of all observations vs respective sample means

SST ≡ Sum of Squared errors Total for all observations vs grand mean = SSA + SSE

MSA ≡ calculated Mean of Sum of All treatment squared errors;

$$E(MSA) = \sigma^2 + \frac{\sum_{i=1}^{k} n_i \alpha_i^2}{k-1}$$

MSE ≡ calculated Mean of Sum of squared Errors; $E(MSE) = \sigma^2$

ANOVA CALCULATIONS:

$$N = \sum_{i=1}^{k} n_i$$

$$\bar{x}_{i\bullet} = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}$$

$$\bar{x}_{\bullet\bullet} = \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} = \frac{1}{N} \sum_{i=1}^{k} n_i \bar{x}_{i\bullet}$$

$$SSA = \sum_{i=1}^{k} n_i (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2$$

$$SSE = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{i\bullet})^2$$

$$SST = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{\bullet\bullet})^2 = SSA + SSE$$

$$MSA = \frac{SSA}{k-1}$$

$$MSE = \frac{SSE}{N-k}$$

$$f = \frac{MSA}{MSE} = \text{value of random variable } F \sim F_{k-1,N-k} \text{ distribution if } H_0 \text{ is true}$$

H_0 rejected if f exceeds F-distribution critical value $f_\alpha(\nu_1{=}k{-}1, \nu_2{=}N{-}k)$

THEORY: From the definition of the α_i, the null and alternate hypotheses, H_0 and H_1, are equivalent to the following statements:

H_0: $\alpha_1 = \alpha_2 = \ldots = \alpha_k$

H_1: At least one $\alpha_i \neq 0$

The MSA and MSE give different estimates of the variance:

$$E(MSA) = \sigma^2 + \frac{\sum_{i=1}^{k} n_i \alpha_i^2}{k-1} \text{ with } k-1 \text{ degrees of freedom}$$

and

$$E(MSE) = \sigma^2 \text{ with } N - k \text{ degrees of freedom}$$

If the null hypothesis is true, the extra term in the MSA estimate is zero since all the α_i are zero. In that case, the ratio of MSA to MSE will have an F-distribution with $k - 1$ and $N - k$ degrees of freedom. We may then use the critical value of the F-distribution from a table to determine if the ratio of MSA to MSE is in the range expected if all the α_i are zero. If the ratio exceeds the critical value, then we may assume that the second term in $E(MSA)$ was not zero after all, and we reject the null hypothesis.

NOTE: Since an F-distribution describes ratios of variances, and variances are always positive, an F-distribution is nonzero only for positive values of f. Thus, critical values of the F-distribution are always positive numbers, and we use a one-sided confidence interval or one-sided hypothesis test in the ANOVA method.

NOTE: The ANOVA method assumes σ is the same for all observations. This might be true, for example, if errors only arise from measurement techniques that are the same for all samples. Bartlett's test is a useful tool for determining if the variances are equal.

NOTE: If we use the same number of observations for all samples (treatments), then the f-ratio is relatively insensitive to small differences in variances, [1].

REF: [1] Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, and Keying Ye, *Probability and Statistics for Engineers and Scientists,* 8th Ed., Upper Saddle River, NJ: Prentice Hall, 2007.

[2] Anthony J. Hayter, *Probability and Statistics for Engineers and Scientists,* 2th Ed., Pacific Grove, CA: Duxbury, 2002.